Current Natural Sciences

NONLINEAR EVOLUTION EQUATIONS

Boling GUO, Yuqin YAO and Lichen ZHAO

Soliton

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This book provides a clear and systematic introduction to solitons, spanning foundational theory, analytical techniques and physical applications. It begins with classical models such as the KdV and nonlinear Schrödinger equations, then develops key elemnts including the inverse scattering transform, Hirota method, Bäcklund transformations, infinite conservation laws, and multidimensional soliton theory. The work explores asymptotic behavior, stability, numerical approaches, and topological structures, while also addressing the geometry of solitons, global-solution problems, and blow-up phenomena. A dedicated chapter examines rogue waves and wave turbulence, and recent advances are highlighted through applications in condensed-matter physics. Balancing rigorous analysis with physical insight, this book serves both as a valuable reference for researchers and graduate students and as a lucid introduction for newcomers to the field.

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Preface

With the development of modern physics and mathematics, the phenomenon of "solitary waves" discovered by the famous British scientist Russell as early as 1834 has attracted great attention and interest in the past decade. It has been confirmed from numerical calculations, theoretical analysis, and physical experiments that a large number of nonlinear evolutionary equations have soliton solutions. The interaction between these solitons exhibits a very peculiar phenomenon, and solitons have stable waveforms. These new phenomena are being utilized to explain some of the new problems that arise in physics. In mathematics, new methods such as scattering inversion, prolongation structure, and Bäcklund transformation have emerged to accurately solve special solutions of nonlinear evolution equations, and have gradually formed a relatively systematic mathematical theory related to soliton problems.

The main purpose of writing this book is to provide a brief introduction to the fundamental problems of solitons, along with their mathematical and physical methods. In addition to the most fundamental knowledge and concepts related to soliton problems, some recent developments and achievements in these research fields were also introduced. We hope that these will help readers sort out some rough but clear threads from the vast works and literatures, so that when readers are interested in a certain aspect of the problem, they can carry out research work based on consulting relevant references.

Below is a brief introduction to the background of the formation of this book. In 1978, one of the authors wrote a delivered notes *KdV Equations and Solitons.* In 1979, at the National Finite Element and Soliton Conference, colleagues such as Feng Kang, Zhang Xueming, Tu Guizhang, and Li Yishen from the conference leadership group believed that there was a need for a book introducing solitons, and entrusted the task of writing this book to the author. This matter also received active support from colleagues from the Science Press. Therefore, based on the "lecture notes", the author began writing, further consulted relevant materials, added content to the original "lecture notes", supplemented the latest literature, and made a new attempt by combining mathematical methods with physical content. However, due to the

extensive knowledge involved in theoretical issues related to solitons and limited for the author's level, there were inevitably inappropriate or erroneous aspects in the book. Readers were kindly requested to correct them.

Finally, we would like to express our special gratitude to colleague Tu Guizhang for his consistent care and enthusiastic assistance in the compilation of this book, as well as for providing many valuable suggestions.

In the process of translation, we strive to maintain the academic value of the original work while taking into account the reading habits and thinking patterns of Chinese readers, and have added a chapter on rogue waves and wave turbulence.

Guo Boling
November 9, 2024

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Chapter 1
Introduction

1.1 The Origin of Solitons

In 1834, British scientist Scott Russell accidentally observed a wonderful water wave. In 1844, he vividly described this phenomenon in his article *On Waves* published in the *Report of the* 14*th Conference of the British Association for the Advancement of Science*: "I observed the movement of a ship, which was pulled by two horses and rapidly advanced along a narrow canal. Suddenly, the ship came to a stop, but the large amount of water pushed by the ship did not stop. They accumulated around the bow of the ship and violently disturbed. Then, the waves suddenly appeared as a round, smooth, and well-defined huge isolated peak, rolling forward at a huge speed and rapidly leaving the bow. Its shape and speed did not change significantly during the journey. I rode on a horse and followed closely to observe that it rolled forward at a speed of about eight to nine miles per hour and maintained its original shape of about 30 feet long and 1 to 1.5 feet high. Gradually, its height decreased. When I tracked for 1 to 2 miles, it finally disappeared into the meandering river channel." This was a peculiar phenomenon observed by Russell, who then believed that this isolated wave was a stable solution to fluid motion and called it "solitary wave". Russell was unable to successfully prove and convince physicists of his argument at the time, thus blaming mathematicians for not being able to predict this phenomenon from known fluid motion equations. Subsequently, the issue of solitary waves sparked widespread debate among many physicists of the time. Until 1895, 60 years later, Korteweg de Vries studied the motion of shallow water waves and established the following shallow water wave motion equation for unidirectional motion under the assumption of long wave approximation and small amplitude

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2}\right) \tag{1.1}$$

here, η is the wave height, l is the water depth, g is the gravitational acceleration, α and σ are constants. They conducted a relatively complete analysis of the solitary wave phenomenon and derived a pulse like solitary wave solution with shape invariance

from Eq.(1.1), which is consistent with Russell's description, thus theoretically proving the existence of solitary waves. However, is this wave stable? Can two solitary waves deform after collision? These questions have not been answered yet. Some people even suspect that Eq.(1.1) is a nonlinear partial differential equation, and the superposition principle of solutions does not satisfy it. After collision, the shape of the two solitary waves may be completely destroyed. This viewpoint has led many people to believe that this type of wave is "unstable", and solitary waves remain buried for a long time until new discoveries are made.

Another question is, do solitary waves like what Russell said appear in other physical fields besides fluid mechanics? In the early 20th century, this was an elusive issue. It was not until the 1950s that a new situation emerged due to the work of Fermi, Pasta and Ulam. They connected 64 particles with nonlinear springs to form a nonlinear vibrating string. Initially, all the energy of these resonators was concentrated in one, while the initial energy of the other 63 was zero. According to classical theory, as long as nonlinear effects exist, there will be phenomena such as energy equalization and ergodicity of states, that is, any weak nonlinear interaction can cause the system to transition from a non-equilibrium state to an equilibrium state. But the actual calculation results surprised them greatly, that the concept of achieving energy balance mentioned above was incorrect. In fact, from Fig.1-1, it can be seen that after a long period of time, almost all the energy returns to its original initial distribution, which is the famous FPU problem. At that time, due to their only examining the frequency space, solitary wave solutions could not be found, so the problem was not properly explained. Later, people regarded the crystal as a chain

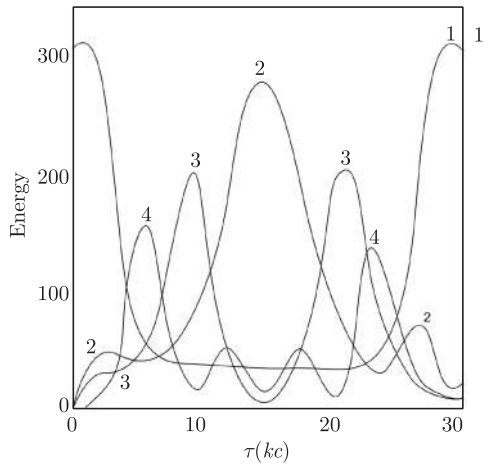


Figure 1-1 Energy curve of FPU problem. The unit is arbitrary, and the initial form of the string is a single sine wave

of elastic tubes with a mass scene and approximately simulated this situation. Toda studied the nonlinear vibration of this mode and obtained solitary wave solutions, which further aroused people's interest in solitary wave research.

Subsequently, in 1962, Perring and Skyrme applied the sine-Gordon equation to the study of elementary particles, and numerical calculations revealed that such solitary waves did not disperse, retaining their original shape and velocity even after a collision.

In 1965, the renowned American scientists Zabusky and Kruskal used numerical simulation methods to investigate in detail the nonlinear interaction process of soliton collisions in plasma, obtaining relatively complete and rich results, and further confirming the theory that soliton interactions do not change the waveform, which surprised people.

Due to the above results and the fact that stable solitary waves with unchanged waveforms after collisions have been discovered in many physical models, many physicists and mathematicians have shown great interest and pay attention in this field. Research on soliton problems boomed and gradually formed a relatively complete soliton theory.

So, what exactly is a "soliton"? Usually, we refer to the local traveling wave solutions of nonlinear evolution equations as "solitary waves". The term "local" denotes that the solutions of differential equations approach zero or a constant as one approaches spatial infinity. We refer to these stable solitary waves, which are solitary waves that do not disappear after colliding with each other and whose waveforms and velocities do not change or only slightly change (just like the common collision of two particles), as "solitons". But there are also several articles and books that confuse solitons with solitary waves.

In physics, solitons are also defined as stable, finite-energy and non-dispersive solutions to classical field equations. That is, if we denote the energy density of a soliton as $\rho(x,t)$, then

$$0 < H = \int \rho(x,t) d^m x < +\infty \ (m \text{ is the dimension of the space})$$

and

$$\lim_{t\to\infty} \max \rho(x,t) \neq 0 \ (\text{for certain } x)$$

That is to say, an soliton can be seen as a finite stable "mass" with non diffusive field energy, which is not destroyed even in motion or collision. For a large number of nonlinear wave equations, their solitons generally take on four shapes, as depicted in Fig.1-2 (a), (b), (c) and (d), namely bell-shape (or wave packet-shape), vortex-shape (anti-bell-shape), kink-shape (knot-shape) and anti-kink-shape (anti-knot-shape).

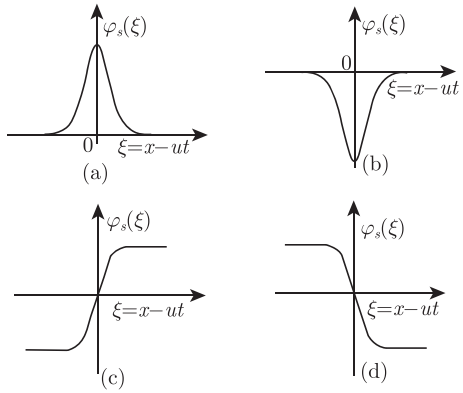


Figure 1-2 Types of solitons

$\varphi_s(\xi)$ represents the traveling wave solution, and $\xi = x - ut$ (u is a constant wave velocity)

Based on the research on elementary particles, Li Zhengdao and others further divided existing solitons into two categories: topological solitons and non topological solitons. We will specifically elaborate on their definitions and detailed information in Chapter 10, and provide a brief introduction to the research work of Li Zhengdao and others on non topological solitons.

1.2 KdV Equation and Its Soliton Solutions

As mentioned above, in 1895, Korteweg and de Vries established the shallow water wave equation (1.1), which we slightly modified to obtain the following form

$$u_t + uu_x + \mu u_{xxx} = 0 \text{ (Add punctuation marks after each formula)} \tag{1.2}$$

here, the constant μ can be positive or negative. If $\mu < 0$, apply the transformation $u \to -u,\ x \to -x,\ t \to t$, then Eq.(1.2) becomes

$$u_t + uu_x - \mu u_{xxx} = 0 \tag{1.3}$$

Thus, it can be assumed that $\mu > 0$. Eq.(1.2) is referred to as the KdV equation.

Let $u(x,t) = u(\xi),\ \xi = x - Dt,\ D = \text{const}$, substituting them into Eq.(1.2) and integrating with respect ξ twice, yields

$$3\mu \left(\frac{du}{d\xi}\right)^2 = -u^3 + 3Du^2 + 6Au + 6B = f(u) \tag{1.4}$$

here, $A,\ B$ are integral constants. The solution of Eq.(1.4) can only be real if $f \geqslant 0$ ($\mu > 0$). If $f(u)$ has only one real root, then it is unbounded. Now let's assume that function $f(u)$ has three real roots, namely $f(u) = -(u - c_1)(u - c_2)(u - c_3)$,

$c_1 < c_2 < c_3$. From this, we can obtain $D = \frac{1}{3}(c_1 + c_2 + c_3)$, $A = \frac{1}{6}(c_1c_2 + c_2c_3 + c_3c_1)$, $B = \frac{1}{6}c_1c_2c_3$. The general form of function $f(u)$ is shown by the curve in Fig.1-3.

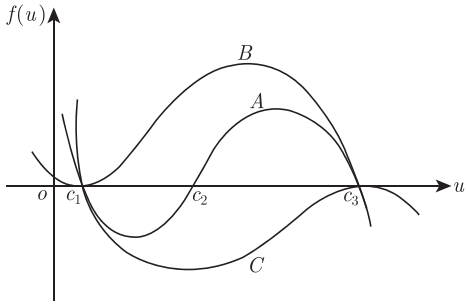


Figure 1-3

The exact solution of Eq.(1.4) can be expressed as a Jacobian elliptic function

$$u = u(x,t) = c_2 + (c_3 - c_2)c_n^2\left[\sqrt{\frac{c_3 - c_2}{12\mu}}\left\{x - \frac{1}{3}(c_1 + c_2 + c_3)t\right\}, k\right] \tag{1.5}$$

here, $k^2 = (c_3 - c_2)/(c_3 - c_1)$. The periodic wave train Eq.(1.5) is usually referred to as the "Cnoidal wave". Since the real period of the function c_n is $2K$ and K is the first type of elliptic integral, the period of the "Cnoidal" wave is $T_p = 4K\sqrt{\frac{3\mu}{c_3 - c_1}}$.

If $K = 0$, $c_n(\xi, 0) = \cos\xi$. Eq.(1.4) has an oscillatory solution

$$u = \bar{c} + a\cos\left[2\sqrt{\frac{c_3 - c_1}{12\mu}}\left\{x - \frac{1}{3}(c_1 + c_2 + c_3)t\right\}\right] \tag{1.6}$$

here $\bar{c} = \frac{c_2 + c_3}{2}$, $a = \frac{c_3 - c_2}{2}$.

If $K = 1$, $c_n(\xi, 1) = \mathrm{sech}\xi$. At this time, the period becomes infinite when $c_2 \to c_1$, as shown in curve B in Fig.1-3. Soliton solution of KdV Eq.(1.2) is obtained

$$u = c_1 + (c_3 - c_1)\mathrm{sech}^2\left[\sqrt{\frac{c_3 - c_1}{12\mu}}\left\{x - \frac{1}{3}(2c_1 + c_3)t\right\}\right] \tag{1.7}$$

Setting $c_1 = u_\infty$, $c_3 - c_1 = a$, then (1.7) becomes

$$u = u_\infty + a\mathrm{sech}^2\left[\sqrt{\frac{a}{12\mu}}\left\{x - \left(u_\infty + \frac{a}{3}\right)t\right\}\right] \tag{1.8}$$

Here, u_∞ represents a uniform state at infinity, and a represents the amplitude of the soliton. From (1.8), it can be seen that the velocity of this solitary wave relative to the uniform state is proportional to the amplitude, while the width of the wave is inversely proportional to the square root of the amplitude, and the amplitude is independent of the uniform state. If $u_\infty = 0,\ \mu = 1$, it can be inferred from (1.8) that

$$u(x,t) = 3D\mathrm{sech}^2\sqrt{\frac{D}{2}}(x - D_t) \tag{1.9}$$

as shown in Fig.1-4.

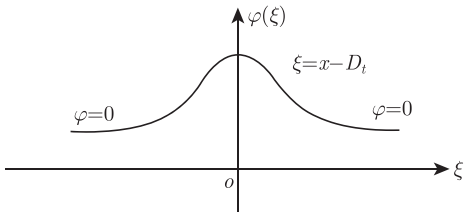


Figure 1-4

Nowadays, it is constantly discovered that a considerable number of wave equations and systems describing weak nonlinear effects can be reduced to the KdV equations under the long wave approximation and small and finite amplitude assumptions. For example: (1) The motion of magnetohydrodynamic waves in cold plasma; (2) Vibration of non resonant lattice; (3) Ionic acoustic waves of plasma; (4) Longitudinal dispersion fluctuations in elastic rods; (5) Pressure wave motion in two mixed states of liquid and gas; (6) The rotation of fluid at the bottom of a pipe; (7) Thermal excitation of phonon wave packets in nonlinear lattices at low temperatures.

1.3 Soliton Solutions for Nonlinear Schrödinger Equations and Other Nonlinear Evolutionary Equations

Cubic nonlinear Schrödinger equations

$$iu_t + u_{xx} + \nu|u|^2 u = 0 \tag{1.10}$$

or in a more general form

$$u_t - \gamma u_{xx} = \chi u - \beta|u|^2 u \tag{1.11}$$

here, $\beta = \beta_0 + i\beta_1, \gamma = \gamma_0 + i\gamma_1, i = \sqrt{-1}, \beta_0, \beta_1, \gamma_0, \gamma_1,\ \chi, \nu$ all are real constants. Such equations have been found in many physics problems, for example, in fine beam

flow (nonlinear optics), there is the following equation

$$2ik\frac{\partial \Psi}{\partial x} + \nabla_{\perp}^2 \Psi + \frac{n_2}{n_0}k^2|\Psi|^2\Psi = 0 \tag{1.12}$$

here, $\nabla_{\perp}^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{m}{r}\dfrac{\partial}{\partial r}$, and $m = 0$ is a plane. $m = 1$ is cylindrical symmetry. $\Psi = ae^{ik\theta}, \theta = kx - wt + ks(x, r), k$ is wave number, and $n = \dfrac{c_0 k}{w} = n_0 + \dfrac{1}{2}n_2 a^2$. For two-dimensional flow, there is an equation

$$\frac{n_2}{n_0}k^2|\Psi|^2\Psi = -2ki\Psi_x - \Psi_{xx} - \Psi_{yy} \tag{1.13}$$

Other examples include Langmuir waves in plasma, self-modulation of one-dimensional monochromatic waves, self-focusing of two-dimensional steady-state plane waves, and the Ginzbug-Landau equation for the motion of superconducting electron pairs in electromagnetic fields can be described using the nonlinear Schrödinger equation.

Now consider the traveling wave solution of Eq.(1.10). Set

$$u(x, t) = e^{irx - ist}v(\xi), \quad \xi = x - Dt$$

where $r,\ s$ are all undetermined constants, and $D = \text{const}$. Substituting the expression for u into (1.10), the ordinary differential equation for v can be obtained

$$v'' + i(2r - D)v' + \left(s - r^2\right)v + \nu|v|^2 v = 0 \tag{1.14}$$

Now taking $r = \dfrac{D}{2}, s = \dfrac{D^2}{4} - \alpha\ (\alpha > 0)$, and eliminating v' (v *is real*) gives

$$v'' - \alpha v - \nu v^3 = 0 \tag{1.15}$$

After integrating, we obtain

$$v'^2 = A + \alpha v^2 - \frac{\nu}{2}v^4 \tag{1.16}$$

When $\nu > 0,\ A = 0$, we have

$$v(x, t) = \left(\frac{2a}{\nu}\right)^{\frac{1}{2}} \operatorname{sech} \alpha(x - Dt) \tag{1.17}$$

Obviously, $|u|^2 \propto \operatorname{sech}^2 \alpha(x - Dt)$, and $v(x, t)$ is called an envelope solitary wave. Now consider a more general form of solution for Eq.(1.10)

$$u(x, t) = \Phi(x, t)e^{i\theta(x,t)} \tag{1.18}$$

where Φ, θ are all real functions and θ is called carrier wave, Φ is called an envelope wave. Substituting Eq.(1.18) into Eq.(1.10) and separating the real part and imaginary part, we can obtain

$$\Phi_{xx} - \Phi\theta_t - \Phi\theta_x^2 - \nu\Phi^3 = 0, \nu > 0 \tag{1.19}$$

$$\Phi\theta_{xx} + 2\Phi_x\theta_x + \Phi_t = 0$$

Setting $\theta = \theta\left(x - D_1 t\right), \Phi = \Phi\left(x - D_2 t\right)$ and substituting them into (1.19) gives

$$\Phi_{xx} + D_1\Phi\theta_x - \Phi\left(\theta_x\right)^2 + \nu\Phi^3 = 0 \tag{1.20}$$

$$\Phi\theta_{xx} + 2\Phi_x\theta_x - D_2\Phi_x = 0 \tag{1.21}$$

For fixed t, regarding (1.21) as a function of x and integrating (1.21) yields

$$\Phi^2\left(2\theta_x - D_2\right) = \varphi(t)$$

Taking $\varphi(t) = 0$ yields $\theta_x = \dfrac{D_2}{2}$. Substituting it into (1.20) and integrating (1.20) with respect to x, we obtain

$$\int_{\Phi_0}^{\Phi} \frac{d\Phi}{\sqrt{P(\Phi)}} = x - D_2 t$$

where

$$P(\Phi) = -\frac{\nu}{2}\Phi^4 + \frac{1}{4}\left(D_2^2 - 2D_1 D_2\right)\Phi^2 + C$$

If $C = 0, D_2^2 - 2D_1 D_2 > 0$, then $\Phi = 0$ is the double root of equation $P(\Phi) = 0$ and there are still two roots $\Phi = \pm\Phi_0$, where $\Phi_0 = \sqrt{\dfrac{D_2^2 - 2D_1 D_2}{2\nu}}$. At this point

$$\Phi = \Phi_0 \operatorname{sech}\left[\sqrt{\frac{\nu}{2}}\Phi_0\left(x - D_2 t\right)\right]$$

as shown in Fig.1-5.

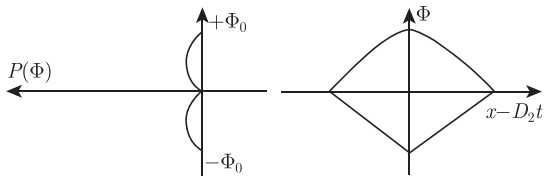


Figure 1-5

If $P(\Phi) < 0$, there cannot be a traveling wave solution. If $C \neq 0$, and if $\left[\frac{1}{4}\left(D_2^2 - 2D_1D_2\right)\right]^2 + 2\nu C \geqslant 0$ or

$$C > -\frac{1}{8\nu}\left(\frac{D_2^2}{2} - 2D_1D_2\right)$$

and $C < 0$, then $P(\Phi)$ has single roots $\pm\Phi_1, \pm\Phi_2$, as shown in Fig.1-6.

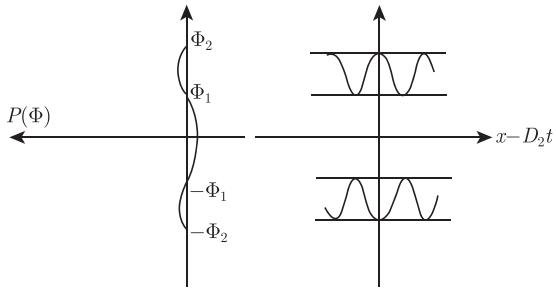


Figure 1-6

Here, the integral is an elliptic function

$$\Phi = \Phi_1\left[1 - \left\{\left(1 - \frac{\Phi_1^2}{\Phi_2^2}\right)\mathrm{sn}^2\left[\frac{\sqrt{\nu}}{2}(x - D_2t)\right]\right\}\right]^{-\frac{1}{2}}$$

where elliptic function sn has the modulus $\mathrm{mod}r = 1 - \frac{\Phi_1^2}{\Phi_2^2}$.

Other well-known equations such as the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0 \tag{1.22}$$

has soliton solution

$$u = 4\tan^{-1} \pm \left\{\pm\left(1 - D^2\right)^{-\frac{1}{2}}(x - Dt)\right\}$$

When both inside and outside the curly braces take a positive sign, it represents a kink from $\Phi = 0(x = -\infty)$ to $\Phi = 2\pi(x = +\infty)$; When both inside and outside the curly braces take a negative sign, it indicates a kink from $\Phi = -2\pi(x = -\infty)$ to $\Phi = 0(x = +\infty)$; When different signs are used inside and outside the curly braces, it indicates an anti-kink, as shown in Fig.1-7.



Figure 1-7

There are many other nonlinear evolution equations with soliton solutions, such as the nonlinear Klein-Gordon equation, Toda nonlinear lattice equation, ferromagnetic chain equation, nonlinear electronic filtering equation, Boussinesq equation, Hirota equation, Born-Infeld equation, etc.

1.4 Experimental Observation and Application of Solitons

The objective existence of solitons has long been confirmed by Russell's observations of water waves. In addition to the soliton image obtained through numerical calculations, it has also been continuously observed in experiments. For example, in the early 1970s, Ikezi, Taylor, Baker, and others witnessed the propagation of KdV type solitons in shallow water waves during water tank experiments. And it is consistent with the results calculated by the computer. In laser shooting, we also observed the propagation of vortex solitary waves caused by collapse and the generation of solitons when the laser beam self focuses in nonlinear media. Recently, L. F. Mollenauer, R. H. Stolen, and J. P. Gordon from the Holmdel Bell Telephone Laboratory in New Jersey, USA, observed the propagation of light pulse type solitons in quartz core fiber materials, which is consistent with the numerical results calculated by J. Satsuma and N. Yajima. In addition, the Josephson effect of superconduction is one of the most important topics in contemporary physics and electronic technology. In the two superconducting materials that make up the Josephson junction, the phase difference φ between the superconducting wave functions satisfies the sine-Gordon equation. The existence of soliton solutions has been confirmed using a superconducting transmission line with a Josephson tunnel junction branching.

The theory of solitons has successfully explained the density pit problem and infrared outward shift problem that have not been explained by classical theory for many years in laser shooting. Recently, Bell Laboratories in the United States have been researching the use of solitons to improve signal transmission systems and increase their transmission rates, which means that they have the advantages of no loss of waveform, no change in speed, high fidelity, and good confidentiality during transmission. They claimed to have made remarkable progress in their experiments on soliton transmission of optical fibers, which undoubtedly promotes the development of research work on soliton theory and provides a more solid foundation.

1.5 Research on the Problem of Soliton Theory

Due to the occurrence of soliton problems in a large number of nonlinear physics fields and their shared characteristics, they have aroused the interest of many physicists. They hope to use soliton theory to study some difficult problems in plasma physics, fundamental quantum physics, and condensed matter physics, as well as the motion laws of matter under nonlinear effects. From a mathematical perspective, it has been discovered that a large class of nonlinear evolutionary equations has soliton solutions, and these nonlinear evolutionary equations with soliton solutions share a series of important and common characteristics. They have infinite conservation laws. There is an analytical method for solving linear equations — inverse scattering

method. There is a Bäcklund transformation. They are completely integrable, etc. Some research has also been conducted on the interrelationships between these important characteristics. There have emerged numerous distinctive branches within mathematical methods. The inverse scattering method, formulated through solving boundary value problems of ordinary differential equations and Gelfand-Marchenko integral equations, has evolved into a useful approach for solving a broad class of nonlinear evolutionary equations, developed by Lax, Zakharov, Shabat and AKNS. Utilizing specific function transformations, including the Bäcklund transform and Hirota transform, a multitude of soliton solutions have been identified. The prolongation structure method, formulated using exterior differential forms and Lie groups, offers a robust tool for solving nonlinear evolutionary equations. To explore the stability and interactions of solitons, a plethora of numerical methods for nonlinear evolutionary equations have been proposed and have achieved significant development. It is no exaggeration to say that the study of soliton theory has emerged as a vibrant field within applied mathematics, attracting numerous mathematicians. Consequently, this research is poised to significantly advance fields like differential equations, functional analysis, group theory, homotopy theory, and topology.Current research on soliton theory has achieved significant advancements, particularly in one-dimensional problems, which are progressing towards greater depth and essentiality. The current development trend is towards multidimensional aspects, with some achievements in multidimensional inverse scattering methods, multidimensional Bäcklund transformations, multidimensional Fourier transforms, multidimensional analytical solutions, and numerical calculations. The second is to explore the development of nonlinear coupled evolutionary equation systems, such as investigating solitons in non fully integrable systems in plasma physics. Although these issues are relatively difficult and there have been few achievements so far, there is still great potential.

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Chapter 2
Inverse Scattering Method

2.1 Introduction

With the formation and development of soliton theory, the inverse scattering method has become an extremely important method for exact solutions of solitons and a rather wide range of a large number of nonlinear evolution equations, and it occupies an important place in soliton theory. The most important feature and advantage of this method is that this rather complex group of nonlinear equations can be solved exactly by combining several linear equations. In the last decade, this method has been mutually promoted and progressively improved with the consequent development of other methods for finding soliton solutions, such as Bäcklund transform, Hirota method, prolongation structure method, etc. This method was first discovered by GGKM in the KdV equation, and later generalized by Lax, Zakharov, Shabat, AKNS and others to a large and very wide range of nonlinear evolutionary equations (which include systems of equations and higher dimensional cases), making them a more general method for linear exact solutions. More recently, good work has been done linking it to the qualitative theory of differential equations, opening up a new avenue for research in the theory of differential equations. In this chapter, we will introduce the important concepts of the inverse scattering method, its most basic elements as well as some recent research results, and present some open problems.

2.2 The KdV Equation and Inverse Scattering Method

We know that using the well-known Hopf-Cole transformation for the Burgers equation

$$u_t + uu_x - \alpha u_{xx} = 0 \ (\alpha > 0) \tag{2.1}$$

i.e., by making

$$u = -2\alpha \frac{w_x}{w} \tag{2.2}$$

it can be reduced to a linear heat conduction equation

$$w_t = \alpha w_{xx} \tag{2.3}$$

for the unknown function w, and the solution of the original equation has the expression

$$\begin{aligned} u(x,t) = &\int_{-\infty}^{\infty} \frac{x-\xi}{t} \exp\left[-\frac{(x-\xi)^2}{4\alpha t} - \frac{1}{2\alpha}\int_0^{\xi} u_0(\xi')d\xi'\right] d\xi \\ &\Big/ \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4\alpha t} - \frac{1}{2\alpha}\int_0^{\xi} u_0(\xi')d\xi'\right] d\xi \end{aligned} \tag{2.4}$$

where $u_0(x)$ is the initial condition, and $u|_{t=0} = u_0(x)$. When $\alpha \to 0$, it can be shown that it converges to a generalized solution of the quasilinear hyperbolic equation

$$u_t + uu_x = 0 \tag{2.5}$$

Is there a transformation similar to (2.2) for the KdV equation? Now consider the KdV equation of the following form

$$u_t - 6uu_x + u_{xxx} = 0 \tag{2.6}$$

Let $u = v^2 + v_x + \lambda$. If $u(x,t)$ is considered as a known function in this equation, it is the Riccati equation for the unknown function $v(x,t)$. Making the transformation $v = \psi_x/\psi$ gives the one-dimensional Schrödinger equation

$$\psi_{xx} - (u - \lambda)\psi = 0 \tag{2.7}$$

where ψ is the wave function, u is the potential and λ corresponds to the energy spectrum. We note that the function u depends not only on x but also on the parameter t. Thus, in general, ψ and λ also depend on the parameter t. We know that if the solution of the KdV equation is smooth, bounded, and tends to zero when $|x| \to \infty$, then in Schrödinger equation (2.7), there are finite discrete spectrums when $\lambda < 0$, given by $\lambda_m = -k_m^2\,(m = 1, 2, \cdots, N)$; $\lambda > 0$ indicates the continuous spectrum $\lambda = k^2\,(-\infty < k < \infty$, k is real). For fixed t, we define the solution of the scattering problem (2.7) to satisfy the boundary conditions

$$\begin{aligned} &\psi(x,k,t) \sim e^{-ikx} + b(k,t)e^{ikx}, \quad x \to +\infty \\ &\psi(x,k,t) \sim a(k,t)e^{-ikx}, \qquad\qquad x \to -\infty \end{aligned} \tag{2.8}$$

and the solution of its bounded state satisfies the boundary conditions

$$\begin{cases} \psi_m(x, k_m(t), t) \sim c_m(k_m(t), t)e^{-k_m x}, & x \to +\infty \\ \psi_m(x, k_m(t), t) \sim e^{k_m x}, & x \to -\infty \end{cases} \tag{2.9}$$

where $b(k,t)$ is called the reflection coefficient, $a(k,t)$ is the penetration coefficient, c_m is the attenuation factor, and

$$\int_{-\infty}^{\infty} \psi_m^2 dx = 1, \quad |a|^2 + |b|^2 = 1 \tag{2.10}$$

as shown in Figure 2-1.

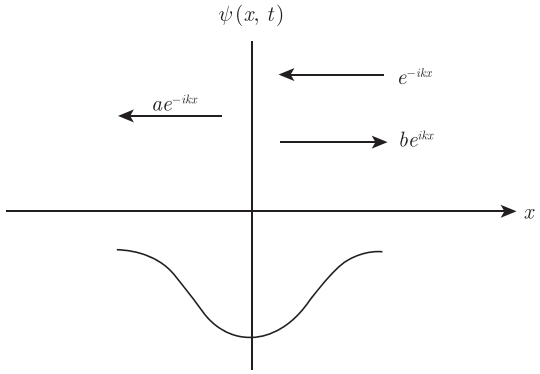


Figure 2-1

The "direct" formulation of the scattering problem is: given the potential u, seek the scattering parameters k_m, c_m, $a(k)$, $b(k)$, and determine the wave function ψ. The inverse scattering problem is now presented as follows: given the scattering parameters k_m, c_m, $b(k)$, determine the potential u. Regarding the connection between the potential u, energy spectrum λ and the scattering parameters in Schrödinger equation (2.7), the following results are obtained:

The potential u of the inverse scattering problem is

$$u(x,t) = -2\frac{d}{dx}K(x,x,t) \tag{2.11}$$

where K satisfies the following GLM integral equation:

$$\begin{gathered} K(x,y,t) + B(x+y,t) + \int_x^{\infty} B(y+z,t)\cdot K(x,z,t)dz = 0 \\ y > x, \quad K(x,z,t) \to 0,\ z \to \infty \end{gathered} \tag{2.12}$$

The kernel of the integral equation (2.12) is

$$B(x,t) = \sum_{m=1}^{N} c_m^2(t)e^{-k_m x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,t)e^{ikx}dk \tag{2.13}$$

where summation denotes to the discrete spectrum and integration denotes to the continuous spectrum. Obviously, the connection between the potential and the scattering parameters pointed out above does not really address the solution of the inverse scattering problem. Because, in order to determine the potential u, K must be determined; in order to determine K, one must determine the scattering parameters c_m, k_m, $b(k)$, etc., which, in turn, depends on the potential u, creating an endless loop. In order to break this loop and actually solve the potential u from the inverse scattering problem (where u is the solution to the KdV equation we require), an important fact is revealed about the KdV equation in relation to the corresponding Schrödinger equation. We have the following theorem:

Theorem 2.1 *Consider the Schrödinger equation*

$$\psi_{xx} - (u - \lambda)\psi = 0, \quad -\infty < x < +\infty$$

If $u(x,t)$ is a solution to the KdV equation and converges sharply to zero when $|x| \to \infty$, then the discrete eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$ of (2.7) are constants (independent of t).

Proof Substituting $u = \dfrac{\psi_{xx}}{\psi} + \lambda$ from (2.7) into the KdV equation (2.6) and multiplying by ψ^2 gives

$$\lambda_t \psi^2 + [\psi R_x - \psi_x R]_x = 0 \tag{2.14}$$

where

$$R \equiv \psi_t + \psi_{xxx} - 3(u + \lambda)\psi_x$$

When $|x| \to \infty$, the eigenfunctions corresponding to the eigenvalues λ_n and their derivatives converge to zero. Integrating (2.14) yields

$$\lambda_{nt} \int_{-\infty}^{\infty} \psi_n^2 dx = 0$$

From the normalization condition $\displaystyle\int_{-\infty}^{\infty} \psi_n^2 dx = 1$, we have

$$\lambda_{nt} = 0, \text{ i.e. } \lambda_n = \text{const}$$ □

The result that λ_n is constant is very important, it means that the inverse scattering problem of the Schrödinger equation only needs to start from the initial value $u_0(x)$ of the KdV equation to determine its discrete eigenvalues and all scattering parameters. Using $\lambda_t = 0$, (2.14) becomes

$$\psi R_{xx} - R\psi_{xx} = 0$$

i.e. $R_{xx}-(u-\lambda)R=0$. It has the exact same form as (2.7), so R can be represented by a linear combination of the eigenfunctions of (2.7), i.e.

$$R\equiv\psi_t+\psi_{xxx}-3(u+\lambda)\psi_x=C\psi+D\varphi \tag{2.15}$$

where C,D depend on t and φ is the solution to the equation (2.7) that is linearly independent of ψ. If take $\varphi=\psi\int_0^x\frac{dx}{\psi^2}$, we have

Theorem 2.2 *Under the condition of Theorem* 2.1, *the scattering parameters of scattering problem* (2.7) *are*

$$\begin{aligned}&c_n(t)=c_n(0)e^{4k_n^3t}\\&b(k,t)=b(k,0)e^{8ik^3t}\\&a(k,t)=a(k,0)\end{aligned} \tag{2.16}$$

where $c_n(0),b(k,0),a(k,0)$ *are determined by the initial value* $u_0(x)$ *of the KdV equation.*

Proof If ψ_n is the eigenfunction, $\varphi_n=\psi_n\int_0^x\frac{dx}{\psi_n^2}$, then φ_n is exponentially unbounded when $x\to+\infty$, so $D(t)=0$ in (2.15). Multiply (2.15) by ψ_n and integrate over infinite intervals to get

$$\int_{-\infty}^{\infty}\left(\frac{1}{2}\psi_n^2\right)_t dx+\int_{-\infty}^{\infty}\left(\psi_n\psi_{nxx}-\frac{3}{2}\psi_{nx}^2-3\lambda\psi_n^2\right)_x dx=c\int_{-\infty}^{\infty}\psi_n^2dx$$

It can be seen from $\int_{-\infty}^{\infty}\psi_n^2dx=1$ and the boundary conditions that the left integrals of the above equation are both zero, so $c(t)\equiv0$. And since $x\to+\infty$, $\psi\sim c_n(t)e^{-k_nx}$, from (2.15) and $u\to0$ $(x\to+\infty)$, we can get

$$c_n^{'}(t)-4k_n^3c_n(t)=0$$

Therefore, $c_n(t)\equiv c_n(0)e^{4k_n^3t}$.

Secondly, for the continuous spectrum, it can be considered that λ is independent of t, so ψ satisfies (2.15), then using the steady radiation condition of the plane wave, that is

$$\psi\sim a(k,t)e^{-ikx},\ \ x\to-\infty$$

substituting it into (2.15) yields

$$(a_t+ik^3a+3k^3a)e^{-ikx}=Ca(k,t)e^{-ikx}+\frac{D}{a}e^{-ikx}\int_0^x e^{2ikx}dx,$$

$$a_t + 4ik^3 a = Ca + \frac{D}{a}\int_0^x e^{2ikx}dx$$

Then we have $D = 0$, and $a_t + (4ik^3 - C)a = 0$. When $x \to +\infty$, substituting $\psi \sim e^{-ikx} + b(k,t)e^{ikx}$ into (2.15), given that the coefficients of the linearly independent function $e^{\pm ikx}$ is zero, we get

$$C = 4ik^3, \;\; b_t - 8ik^3 b = 0, \;\; b(k,t) = b(k,0)e^{8ik^3 t}$$

From $C = 4ik^3$, we have $a_t = 0$, i.e., $a(k,t) = a(k,0)$. □

Theorem 2.1 and Theorem 2.2 provide a concrete way to solve the initial value problem of KdV equation

$$\begin{cases} u_t - 6uu_x + u_{xxx} = 0, \;\; -\infty < x < \infty, \;\; t > 0 \\ u(x,0) = u_0(x) \end{cases}$$

by Schrödinger equation and scattering problem. First, we solve the eigenvalue problem

$$\psi_{xx} - [u_0(x) - \lambda]\psi = 0 \tag{2.17}$$

For this problem, the scattering parameters k_n, $c_n(0)$, $b(k,0)$ are determined, then $c_n(t)$, $b(k,t)$ is obtained by (2.16), which leads to

$$\begin{aligned} B(x+y,t) &= \sum_{n=1}^{N} c_n^2(t)e^{-k_n(x+y)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,t)e^{ik(x+y)}dk \\ &= \sum_{n=1}^{N} c_n^2(0)e^{8k_n^3 t - k_n(x+y)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,0)e^{i[8k^3 t + k(x+y)]}dk \end{aligned}$$

Then $K(x,y,t)$ is determined by GLM equation

$$K(x,y,t) + B(x+y,t) + \int_x^{\infty} B(y+z,t)K(x,z,t)dz = 0 \;\; (y > x)$$

Thus, we obtain the solution

$$u(x,t) = -2\frac{d}{dx}K(x,x;t)$$

The solution steps are shown in Figure 2-2.

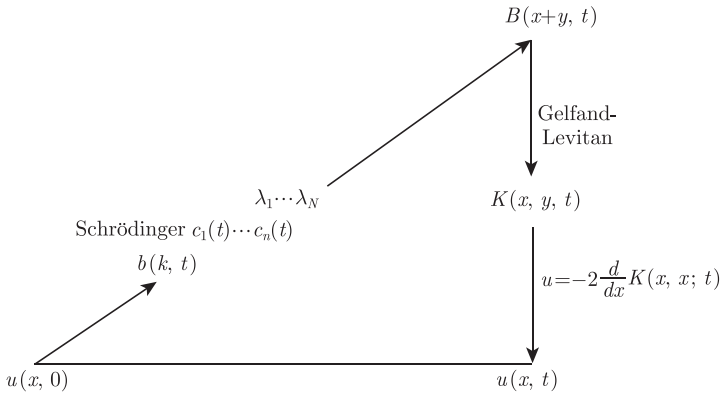


Figure 2-2

In this way, we turn an initial value problem for solving a nonlinear equation (KdV equation) into a problem for solving two linear equations, one is the Sturm-Liouville problem for second-order ordinary differential equation, and the other is solving a linear integral equation. Let's take two examples to illustrate this.

Let $u_0(x) = -2\operatorname{sech}^2 x$. The eigenvalue problem corresponding to (2.17) which we can solve precisely with hypergeometric functions, has a discrete value $k_1 = 1$, the corresponding normalization constants $c_1(0) = \sqrt{2}$, $b(k,0) = 0$, and therefore $b(k,t) \equiv 0$, $\forall t \geqslant 0$. The corresponding Gelfand-Levitan equation is

$$K(x,y,t) + 2e^{8t-x-y} + 2e^{8t-y}\int_x^\infty K(x,z,t)e^{-z}dz = 0$$

Let $K(x,y,t)$ be a separable variable[①], i.e. $K(x,y,t) = L(x,t)e^{-y}$, then we obtain that

$$L(x,t) + 2e^{8t-x} + 2e^{8t}L(x,t)\int_x^\infty e^{-2z}dz = 0$$

$$L(x,t) = \frac{-2e^x}{1+e^{2x-8t}},\quad K(x,y,t) = \frac{-2e^{x-y}}{1+e^{8x-8t}}$$

It is easy to verify that $K(x,y,t)$ determined in this way does satisfy the Gelfand-Levitan equation and is the unique solution. Therefore, the solution of KdV equation is

$$u(x,t) = \frac{8e^{2x-8t}}{(1+e^{2x-8t})^2} = -2\operatorname{sech}^2(x-4t)$$

① This is generally not possible for non separable variables, where $K(x,y,t)$ should be extended as a series.

which, as can be seen from the travelling wave solution, is indeed the exact solution of the initial value problem of KdV equation.

Another example is taking $u(x,0) = u_0(x) = -6\,\mathrm{sech}^2 x$, then two different eigenvalues $k_1 = 2$ and $k_2 = 1$ can be solved. From $b(k,0) = 0$, the solution of KdV equation can be obtained as

$$u(x,t) = -12\,\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}$$

Next, we consider the problem of solving N solitons by inverse scattering method. In this case, the reflection coefficient $b(k,t) \equiv 0$, which is only the case of the discrete spectrum. Then the Gelfand-Levitan equation is

$$K(x,y,t) + \sum_{m=1}^{N} c_m^2(t) e^{-k_m(x+y)} + \sum_{m=1}^{N} c_m^2 e^{-k_m y} \int_x^{\infty} e^{-k_m t} K(x,z,t)dz = 0 \quad (2.18)$$

where $c_m = c_m(t) = c_m(0)e^{4k_m^3 t}$, and $k_m > 0$ are different. Let $K(x,y,t)$ have the form

$$K(x,y,t) = -\sum_{m=1}^{N} c_m \psi_m(x) e^{-k_m y} \quad (2.19)$$

where ψ_m is the undetermined function and c_m is the normalized factor. Substituting (2.19) into (2.18) and taking the coefficient of $e^{k_m y}$ as 0, we get that $\psi_m(x)$ should satisfy the following linear algebraic equation

$$\psi_m(x) + \sum_{m=1}^{N} c_m c_n \frac{e^{-(k_m+k_n)x}}{k_m + k_n} \psi_n(x) = c_m e^{-k_m x} \quad (m = 1, 2, \cdots, N) \quad (2.20)$$

If we use the matrix representation

$$I \equiv (\delta_{mn}), \quad C \equiv \left(c_m c_n \frac{e^{-(k_m+k_n)x}}{k_m + k_n} \right), \quad \psi = (\psi_1, \psi_2, \cdots, \psi_N)^{\mathrm{T}}$$

$$E = (c_1 e^{-k_1 x}, c_2 e^{-k_2 x}, \cdots, c_N e^{-k_N x})^{\mathrm{T}}$$

then (2.20) can be written as

$$(I + C)\psi = E \quad (2.21)$$

For the solution ψ in equation (2.20) to exist, it is sufficient to prove that matrix C is positive definite. In fact, there is

$$\sum_{m=1}^{N} \sum_{n=1}^{N} p_m p_n c_m c_n \frac{e^{-(k_m+k_n)x}}{k_m + k_n} = \int_x^{\infty} \left[\sum_{m=1}^{N} p_m c_m e^{-k_m z} \right]^2 dz > 0$$

So we know that $I+C$ is positive definite, and thus (2.20) has a unique solution. We can solve for ψ using Cramer's rule. Let Q_{mn} be the algebraic cofactor of an element a_{mn} in matrix $I+C$, and expand on the nth row to get

$$\Delta \equiv \det(I+C) = \sum_m \left(\delta_{mn} + c_m c_n \frac{e^{-(k_m+k_n)x}}{k_m+k_n}\right) Q_{mn}$$

$$\psi_n(x) = \Delta^{-1} \sum_m c_m e^{-k_m x} Q_{mn}$$

From (2.19), $y=x$, there is

$$\begin{aligned} K(x,y,t) &= -\sum c_n \psi_n(x) e^{-k_n x} \\ &= -\Delta^{-1} \sum_m \sum_n c_m c_n e^{-(k_m+k_n)x} Q_{mn} \\ &= \Delta^{-1} \frac{d}{dx} \Delta \end{aligned}$$

Therefore, the potential without reflection coefficient is

$$u(x,t) = -2\frac{d^2}{dx^2} \log\det(I+C) \tag{2.22}$$

which is the solution to the KdV equation.

In the Gelfand-Levitan integral equation, due to the symmetric position of K and B, the method of determining B by the scattering parameter first and then solving K may not be adopted, instead, the solution K satisfying the linear hyperbolic equation is used to solve B by the Gelfand-Levitan equation, thus forming another solution to the scattering inversion problem. The solution block diagram is shown in Figure 2-3.

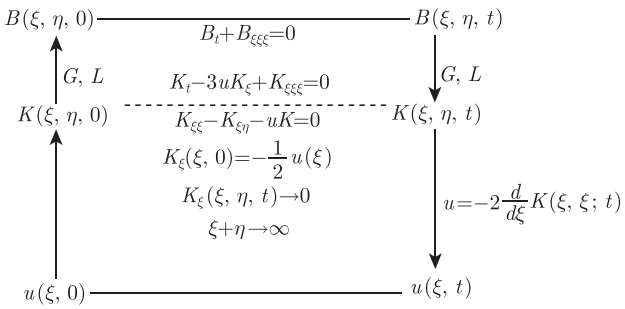


Figure 2-3

2.3 Lax Operator and Generalization of Zakharov, Shabat, AKNS

Consider the general nonlinear evolution equation

$$u_t = K(u) \tag{2.23}$$

where $K(u)$ represents a nonlinear operator on some appropriate function space. If we can find linear operators L and B, which depend on the solution u of the equation (2.23) and satisfy the operator equation (commonly known as the Lax operator equation)

$$iL_t = BL - LB = [B, L] \ \ (i = \sqrt{-1}) \tag{2.24}$$

where B is a self-adjoint operator, then the eigenfunction ψ corresponding to the eigenvalue E of the operator L can be derived from (2.24), i.e.

$$L\psi = E\psi \tag{2.25}$$

If the change of ψ with time satisfies the equation

$$i\psi_t = B\psi \tag{2.26}$$

then E does not change with time. In fact, differentiating (2.25) by t and multiplying by i yields

$$\begin{aligned} i\left[\psi\frac{dE}{dt} + E\frac{d\psi}{dt}\right] &= i\left[L\psi_t + \frac{\partial L}{\partial t}\psi\right] \\ &= iL\psi_t + [BL - LB]\psi = L(i\psi_t - B\psi) + EB\psi \end{aligned}$$

From (2.26), $i\psi_t = B\psi \Rightarrow i\psi\dfrac{dE}{dt} = 0$.

To solve the initial value problem of the given equation (2.23), that is, given $u(x, 0)$, find the solution $u(x, t)$ that satisfies (2.23), generally through the following steps:

(i) Find solution to the direct problem. That is, the scattering parameter of ψ at $|x| = \infty$ (such as the eigenvalue of the operator L, reflection, penetration coefficient, etc.) is calculated from the known value $u(x, 0)$ of $t = 0$.

(ii) Find the change of scattering data with time. From (2.26), considering the asymptotic form of B at $|x| = \infty$, the variation of scattering data with time is calculated.

(iii) Find the solution to the inverse problem. From the known scattering data of L as a function of time, the solution $u(x, t)$ is constructed.

The steps of solving the general nonlinear evolution equations are simply shown in Figure 2-4.

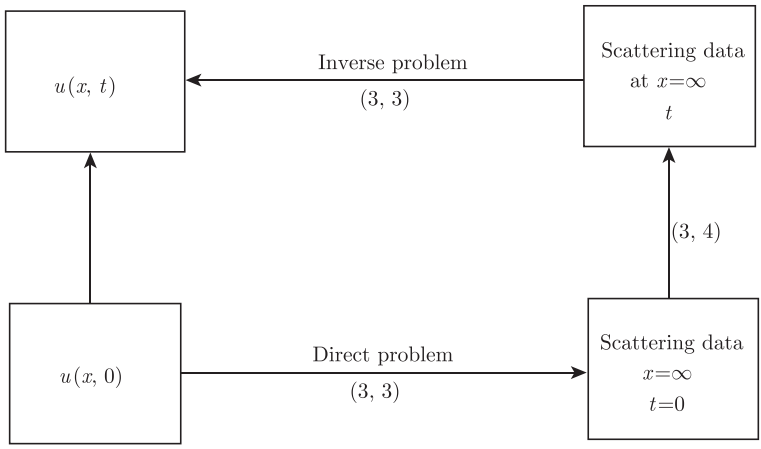


Figure 2-4

Example Consider the solution of KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{2.27}$$

with the initial condition

$$u(x,0) = u_0(x) \tag{2.28}$$

Take the family of the second-order differential operators $L(t)$:

$$L(t) \equiv -\frac{\partial^2}{\partial x^2} + u(x,t) \tag{2.29}$$

where t is the parameter, $u(x,t)$ is the solution of (2.27), and $u(\cdot,t) \in L_2$.

The eigenvalue problem of the operator L is

$$-\psi_{xx} + u(x,t)\psi = E\psi \tag{2.30}$$

As described in section 2.2, we can find the solution to its direct problem, that is, given $u_0(x)$, the scattering parameters ($k_n,\ c_n,\ n = 1,2,\cdots,N;\ a(k),\ b(k),\ 0 \leqslant k^2 < \infty$) can be sought.

Then select the self-conjugation operator

$$B \equiv -4i\frac{\partial^3}{\partial x^3} + 3i\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}\cdot u\right) \tag{2.31}$$

where $u(x,t)$ satisfies the KdV equation (2.27). Then the operators L, B satisfy the operator equation (2.24). From (2.26) and the boundary condition that ψ at infinity should satisfy, $u(x,t) \to 0$ ($|x| \to \infty$), we can easily obtain the evolution of scattering data with time, that is, the result of theorem 2.2. Then from the Gelfand-Levitan integral equation, the solution $u(x,t)$ of the initial value problem of the KdV equation can be obtained.

Obviously, not all nonlinear equations (2.23) can be solved by inverse scattering method. The main difficulty is that it is not always possible to find suitable operators L and B to satisfy the operator equation (2.24) under such conditions, and sometimes for a given L, even if B is found, B may be trivial. For example, $L = \dfrac{\partial^2}{\partial x^2} + u(x,t)$, taking $B = i\dfrac{\partial}{\partial x}$, we have $L_t = u_t$, $[B,L] = i[D,u] = iu_x$, $iL_t = [B,L] \Rightarrow u_t = u_x$. This is a travelling wave equation with an expression $u = f(x-t)$ with an obvious solution. However, despite these problems, it has now been found that a wide range of nontrivial nonlinear wave equations can be solved using the inverse scattering method. The following describes the extensive generalization made by Zakharov, Shabat and AKNS.

Consider the linear problem

$$Lv = \zeta v \tag{2.32}$$

where

$$L \equiv \begin{pmatrix} i\dfrac{d}{dx} & -iq(x,t) \\ ir(x,t) & -\dfrac{d}{dx} \end{pmatrix}$$

$$v \equiv \begin{pmatrix} v_1(x,t) \\ v_2(x,t) \end{pmatrix} \tag{2.33}$$

$q(x,t)$, $r(x,t)$ are arbitrary differentiable functions, and ζ is a constant. Let B have the general form

$$B = \begin{pmatrix} a(x,t;\zeta) & b(x,t;\zeta) \\ c(x,t;\zeta) & -a(x,t;\zeta) \end{pmatrix} \tag{2.34}$$

where the coefficients (a, b, c) are also arbitrary (Initially, Zaharov, Shabat and AKNS consider the case $r = -q^*$). $i\psi_t = B\psi$ takes the form

$$i\frac{dv}{dt} = Bv \tag{2.35}$$

Now write (2.32) in scalar form

$$\begin{cases} iv_{1x} - iqv_2 = \zeta v_1 \\ irv_1 - iv_{2x} = \zeta v_2 \end{cases} \tag{2.36}$$

(2.35) is also written in scalar form

$$\begin{cases} iv_{1t} = av_1 + bv_2 \\ iv_{2t} = cv_1 - av_2 \end{cases} \tag{2.37}$$

Differentiating (2.36) with respect to t gives

$$\begin{cases} iv_{1xt} - iq_t v_2 - iqv_{2t} = \zeta v_{1t} \\ ir_t v_1 + irv_{1t} - iv_{2xt} = \zeta v_{2t} \end{cases} \tag{2.38}$$

Differentiating (2.37) with respect to x gives

$$\begin{cases} iv_{1xt} = a_x v_1 + av_{1x} + b_x v_2 + bv_{2x} \\ iv_{2xt} = c_x v_1 + cv_{1x} - a_x v_2 - av_{2x} \end{cases} \tag{2.39}$$

Eliminating v_{1xt}, v_{2xt} from (2.38), (2.39) yields

$$\begin{cases} a_x v_1 + av_{1x} + b_x v_2 + bv_{2x} = \zeta v_{1t} + iq_t v_2 + iqv_{2t} \\ c_x v_1 + cv_{1x} - a_x v_2 - av_{2x} = ir_t v_1 + irv_{1t} - \zeta v_{2t} \end{cases} \tag{2.40}$$

Then substitute (2.36), (2.37) into (2.40) and eliminate v_{1x}, v_{2x}, v_{1t}, v_{2t} to get

$$\begin{aligned} & a_x v_1 + a\left(-i\zeta v_1 + qv_2\right) + b_x v_2 + b\left(i\zeta v_2 + rv_1\right) \\ = & -\zeta i\left(av_1 + bv_2\right) + iq_t v_2 + q\left(cv_1 - av_2\right) \end{aligned} \tag{2.41}$$

$$\begin{aligned} & c_x v_1 + c\left(-i\zeta v_1 + qv_2\right) - a_x v_2 - a\left(i\zeta v_2 + rv_1\right) \\ = & ir_t v_1 + r\left(av_1 + bv_2\right) + i\zeta\left(cv_1 - av_2\right) \end{aligned} \tag{2.42}$$

Comparing the coefficients of v_1, v_2 for the equation (2.41) gives

$$\begin{aligned} & v_1 : a_x + br = qc, \ \ \therefore a_x = qc - br \\ & v_2 : b_x + 2i\zeta b = iq_t - 2aq \end{aligned}$$

By comparing the coefficients of v_1, v_2 in equation (2.42), we get

$$\begin{aligned} & v_1 : c_x - 2i\zeta c = ir_t + 2ar \\ & v_2 : cq - a_x = br \end{aligned}$$

Thus, we obtain the equations that any coefficient (a, b, c, r, q) should satisfy

$$\frac{\partial a}{\partial x} = qc - rb \tag{2.43}$$

$$\frac{\partial b}{\partial x} + 2i\zeta b = i\frac{\partial q}{\partial t} - 2aq \tag{2.44}$$

$$\frac{\partial c}{\partial L} - 2i\zeta c = i\frac{\partial r}{\partial t} + 2ar \tag{2.45}$$

In the future, for the sake of writing uniformity, we order

$$\frac{dv}{dt} = Mv, \quad M = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$$

We can rewrite the equations (2.43), (2.44), (2.45) as

$$A_x = qC - rB \tag{2.46}$$

$$B_x + 2i\zeta B = q_t - 2Aq \tag{2.47}$$

$$C_x - 2i\zeta C = r_t + 2Ar \tag{2.48}$$

where $A \equiv A(x,t;\zeta)$, $B \equiv B(x,t;\zeta)$, $C \equiv C(x,t;\zeta)$, $r = r(x,t)$, $q = q(x,t)$.

Equations (2.36), (2.37) and (2.46)-(2.48) become the basis of analysis of the inverse scattering method. For the given initial values $r(x,0)$, $q(x,0)$, the equation set (2.36) are used to find the scattering data determined by the discrete eigenvalue (which does not change with time) and the initial time eigenfunction $v_1(x,0;\zeta)$, $v_2(x,0;\zeta)$ at $|x| \to \infty$. If given a set of r_t, q_t, r, q, in principle, we can find the solution A, B, C of the system of equations (2.46)-(2.48). With A, B, C, we can calculate the change of the asymptotic form ($|x| \to \infty$) of the characteristic functions v_1, v_2 with time from the system of equations (2.37), and then find the potential $r(x,t)$, $q(x,t)$ at the next time. Of course, since r, q are unknown, we cannot yet construct them so simply, but it does give us information on how to construct an exact solution to the evolution equation using the inverse scattering method.

First, we look for some very simple solutions for (2.46)-(2.48). Although it is very special, a wide range of nonlinear evolution equations can be obtained from it. If

$$A = \sum_{n=0}^{N} A^{(n)}\zeta^n, \quad B = \sum_{n=0}^{N} B^{(n)}\zeta^n, \quad C = \sum_{n=0}^{N} C^{(n)}\zeta^n \tag{2.49}$$

it is easy to obtain that $A^{(N)} = a_N$ (a_N is independent of x and can depend on t), $B^{(N)} = C^{(N)} = 0$. We first find $A^{(N-1)}$ from (2.46) and then find $B^{(N-1)}$ and $C^{(N-1)}$