Textbooks for Tomorrow's Scientists

ENGINEERING MECHANICS

Dixiong YANG, Junfeng GU, Lei YANG,
Jingjie CHEN and Xiaofei HU

Structural Analysis

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Main features of this textbook lie in: (1) strengthened the interestingness and readability, and increased brief introduction on the developmental history of structural analysis and the important figures; (2) adopted the kinematic method to construct exactly and rapidly the influence lines of forces of statically indeterminate structures proposed by the author, and highlighted the energy principles and methods; (3) increased introducing the backgrounds of engineering applications; (4) from the viewpoints of history, methodology, aesthetic appreciation and creative thinking, inspected structural analysis and strived to cultivate the innovative talents.

The present book emphasizes fundamental theories, concepts, computational methods and engineering applications of structural analysis. It can be used as a textbook for undergraduates with majors of civil engineering, engineering mechanics, flight vehicle design, mechanical engineering, and ocean engineering etc., while being a helpful reference for engineers and professionals in the relevant fields.

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Brief Introduction to the Book

Structural Analysis is a basic course for undergraduate students with majors in civil engineering, engineering mechanics, flight vehicle design, mechanical engineering, naval architecture, ocean engineering, etc., and is also an introductory course for undergraduates to learn and master the analysis and design of beam, truss, frame, arch and composite structures for buildings, bridges and flight vehicles and so on. This textbook includes eight chapters and covers an introduction, kinematic analysis of plane member systems, analysis of statically determinate structures, the principle of virtual work and deflection calculation, force method, displacement method, influence lines of structures under moving loads, and matrix displacement method.

The present book emphasizes the fundamental theories, concepts, computational methods, and engineering applications of structural analysis. It can be used as a textbook for undergraduates with majors in civil engineering, engineering mechanics, flight vehicle design, mechanical engineering, naval architecture, ocean engineering, etc., and also is a helpful reference for associated engineers and professionals.

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Preface

Structural Analysis is a basic course for undergraduate students with majors in civil engineering, engineering mechanics, flight vehicle design, mechanical engineering, naval architecture, ocean engineering, etc., and is also an introductory course for undergraduates to learn and master the analysis and design of various beam, truss, frame, arch and composite structures for buildings, bridges and flight vehicles and so on. This textbook includes eight chapters and covers an introduction (chapter 1), kinematic analysis of plane member systems (chapter 2), analysis of statically determinate structures (chapter 3), principle of virtual work and deflection calculation (chapter 4), force method (chapter 5), displacement method (chapter 6), influence lines of structures under moving loads (chapter 7), and matrix displacement method (chapter 8). Chapters 5, 6, and 8 focus on addressing the analysis of statically indeterminate structures.

The present textbook emphasizes the fundamental theories, concepts, computational methods, and engineering applications of structural analysis. During the preparation of this book, we referred to many excellent textbooks and made efforts to renew some contents and reflect the time trends of digitalization and informatization. The main features of this textbook lie in the following aspects: (1) strengthened the interestingness and readability, and increased the brief introduction on the developmental history of structural analysis and the important figures; (2) substituted the kinematic method to construct exactly and rapidly the influence lines of forces of statically indeterminate structures proposed by the authors for the corresponding old knowledge, and highlighted the energy principle and method; (3) increased introducing the backgrounds of engineering applications for some theories and methods of structural analysis; (4) from the viewpoints of history, methodology, aesthetic appreciation, and creative thinking, inspected the structural analysis and strived to integrate the knowledge, capability and quality education to cultivate the innovative talents.

The main contributors to the contents of this book are Dixiong Yang (chapters 1 and 7), Junfeng Gu (chapters 2 and 3), Lei Yang (chapters 4 and 5), Jingjie Chen (chapter 6), and Xiaofei Hu (chapter 8).

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We will appreciate comments and suggestions from the readers to improve the current edition. All constructive criticisms are welcome with gratitude.

Authors
January 2023

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Notations

A	area
b	width
c	generalized displacement of the support
d	panel
E	elastic modulus
f	arch height
F	force or generalized force
F_{P}	concentrated load
F_{H}	horizontal thrust
F_x, F_y	components of force in horizontal (x) and vertical (y) directions
F_{N}	axial force
$F_{\mathrm{N}x}$, $F_{\mathrm{N}y}$	components of axial force in horizontal (x) and vertical (y) directions
F_{S}	shear
$F_{\mathrm{S}}^{\mathrm{L}}$, $F_{\mathrm{S}}^{\mathrm{R}}$	shears at the left and right sides of section
$F_{\mathrm{S}}^{\mathrm{F}}$	fixed end shear
F_{R}	generalized reaction or reaction resultant
$\overline{\boldsymbol{F}}^e$	force vector of elemental member end in local coordinate system
$\boldsymbol{F}^e$	force vector of elemental member end in global coordinate system
$\overline{\boldsymbol{F}}_{\mathrm{P}}^e$	fixed-end force vector of the element in local coordinate system
G	shear modulus
h	height
i	bending linear stiffness
I	sectional moment of inertia
$\boldsymbol{I}$	identity matrix
k	stiffness coefficient or nonuniform coefficient of shear stress distribution
$\overline{\boldsymbol{k}}^e$	elemental stiffness matrix in local coordinate system
$\boldsymbol{k}^e$	elemental stiffness matrix in global coordinate system

$\boldsymbol{K}$	structural stiffness matrix
l	length or span
m	mass or distributed bending moment
M	moment, couple moment or bending moment
M^{F}	fixed-end moment
n	degree of static indeterminacy
p	intensity of uniformly distributed load in horizontal direction
$\boldsymbol{P}^e$	nodal load vector of element
$\boldsymbol{P}$	nodal load vector of structure
q	intensity of uniformly distributed load in vertical direction
R	radius
r	radius or influence coefficient of reaction
S	static moment
t	temperature
$\boldsymbol{T}$	coordinate transform matrix
U	strain energy
u	horizontal displacement
v	vertical displacement or deflection
w	vertical displacement
W	work, computational degree of freedom, or weight
W_{e}	external virtual work
W_{i}	internal virtual work
X	generalized unknown force or generalized redundant unknown force
y	displacement
Z	response for influence line
α	linear expansion coefficient
Δ	generalized unknown displacement
$\boldsymbol{\Delta}$	displacement vector
$\boldsymbol{\Delta}^e$	nodal displacement vector of elemental member end
δ	flexibility coefficient or displacement influence coefficient
ε	linear strain
μ	Poisson ratio
κ	curvature
φ	angular displacement or chord angle
γ_0	mean shear strain
θ	rotational angle of section
$\boldsymbol{\xi}$	elemental locating vector
ρ	material density
Π	total potential energy

Chapter 1

Introduction

Humans had built various kinds of buildings and structures since ancient times, such as the Pyramids in Egypt, the Gard Aqueduct in France, the Great Wall, Dujiangyan Irrigation System, the Zhaozhou Bridge, and the Forbidden City in China. With the advancement of science and technology, people gradually possessed some knowledge of the rules of structural design and the strength, stiffness, and stability of structures. According to the experience and experiments, a specialized discipline of structural analysis or structural mechanics was formed, which has been widely applied in practical engineering fields.

As far as the basic principles and methods are concerned, structural mechanics was developed simultaneously with theoretical mechanics and mechanics of materials. Therefore, structural mechanics was merged with theoretical mechanics and mechanics of materials in the early stage of development. In the early 19th century, owing to the development of emerging industries, people began to design a variety of large-scale engineering structures that require more accurate analysis and calculation. As a result, the theories and methods for analysis of engineering structures began to be independent. By the middle of the 19th century, structural mechanics started to become an independent discipline. Since the 20th century, owing to the emergence of new materials, new structures, new theories, and methods, the discipline of structural mechanics has been deepening in connotation and expanding in scope, which has developed into a disciplinary group covering the statics, dynamics, stability theory, computational structural mechanics, intelligent material structures, structural control, and health monitoring. Its application scope has also been extended to the fields of civil engineering, hydraulic engineering, mechanical engineering, aerospace engineering, naval architecture, ocean engineering, and so on. In fact, structural analysis is an ancient and evergreen discipline of applied mechanics, which is developing constantly with the times, and is a bridge linking the mechanics theory with engineering practice.

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1.1 Research Object and Tasks of Structural Analysis

1.1.1 Research Object

Engineering structure means the part of the building, construction, or other civil infrastructure that bears and transmits the loads, which plays the role of the skeleton and is simply called a **structure**. For example, the beam-column system in buildings, the dams and gates in hydraulic structures, the highway and railway bridges, tunnels, culverts, and the skeletons of ships, automobiles, airplanes, and manned spacecraft subjected to the loadings, etc., are all typical examples of engineering structures (figure 1.1).



FIG. 1.1 – Building, bridge, and airplane.

The mechanical properties and load-bearing capacities of structures are closely related to their geometric characteristics. The structures can be divided into three categories according to geometric features.

(1) **Member structure**—is formed by some interconnected members. The geometric feature of the member is that the cross-section dimension is much smaller than the length. A beam, arch, truss, and frame are typical forms of the member structure.
(2) **Plate and shell structures**—are also known as thin-walled structures (figure 1.2a, b). Their geometric feature is that the thickness is much smaller than the length and width. The floor slab and shell roof in the building as well as the outer shell of the aircraft and the ship etc., all belong to the plate and shell structures.
(3) **Massive structure**—also called a three-dimensional continuum structure. Its geometric feature is that the dimensions in the length, width, and height are similar. Gravity retaining walls (figure 1.2c) and gravity dams in hydraulic engineering are massive structures.

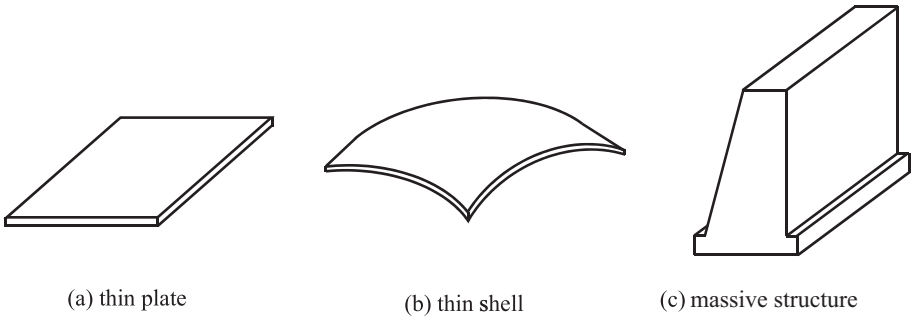


FIG. 1.2 – Thin-walled structure and massive structure.

In a narrow sense, the structure often refers to the member structure, and the usual structural analysis means the analysis of member structure.

Structural mechanics is closely related to theoretical mechanics, mechanics of materials, and elastoplastic mechanics. Theoretical mechanics mainly discusses the basic law of mechanical motion of particle (system) and rigid body (system), ignores the deformation effects of objects, and often regards the object as a particle or rigid body. The other three kinds of mechanics focus on the deformation effect of the object, and discuss the strength, stiffness, stability, and dynamic responses of structures and their members. Among them, the main research object is a single member in the mechanics of materials, and it is the system of member structure in structural analysis, while in elastoplastic mechanics the research objects are massive structure, plate, and shell structures.

1.1.2 Tasks

Structural analysis is a discipline that studies the rational forms of structures (such as beams, trusses, frames, arches, etc.) and the rules or laws of internal forces, deformations, dynamic responses, and stability of structures under external forces, and other external factors. The research purpose is to make the structure meet the requirements of safety, applicability, and economy. Specifically, the basic tasks of structural analysis include the following aspects.

(1) Study the construction rules, mechanical properties, reasonable forms of the structures, and the reasonable choice of structural computational models.
(2) Investigate the computational methods of internal forces and deformations of structures, and check the structural strength and stiffness.
(3) Research the stability of structures and the structural responses and vibration control under dynamic loads.

The research methodology for problems of structural analysis includes theoretical analysis, numerical calculation, and experimental study. The course of structural analysis focuses on theoretical analysis and numerical calculation. In structural analysis, the actual structure is firstly simplified as a computational model, and then

the corresponding computation is performed. There are various kinds of computational approaches in structural analysis. These approaches take into account the following three types of fundamental equations.

(1) Equilibrium equation or motion equation of the structural system.
(2) Geometric continuous equation of deformation (or called compatibility equation).
(3) Constitutive equation between stress and deformation (or called physical equation).

The basic solution method of structural analysis directly utilizes the above three types of fundamental equations for calculation and analysis, which can be called the **equilibrium-geometry-constitutive method**, for example, the force method and displacement method. The solution method is named an **energy method** such as the unit load method, or Rayleigh–Ritz method for displacement computation, if it is expressed with the form of virtual work or energy.

In the past, structural analysis problems were relatively simple and could be solved by analytical approaches and hand computation. With the advent and popularity of electronic computers, a large number of complex engineering analyses and design problems mainly relied on computer calculation. Computer calculation has enhanced the capability of solving problems and also has put forward a new requirement for structural analysis, that is, the computer method must adapt to the attribute of computer calculation. Therefore, some contents closely related to computer analysis of structures, such as energy principle, matrix displacement method, finite element method, discrete element method, meshless method, isogeometric analysis, structural analysis software, and structural optimum design, etc., have occupied an increasingly important position in structural analysis. In the field of structural analysis, energy principles, numerical methods, and program software have formed a new subdiscipline—computational structural mechanics, which is a branch of solving structural analysis and design problems by numerical methods with the aid of computers.

1.2 Computational Models of Structures

The engineering structures are usually very complicated, and it is neither possible nor necessary to perform mechanical analysis completely according to the actual situations. Thus, before implementing the mechanical analysis and structural calculation, the real structure is represented by an abstract and simplified model, which is called a **computational model of structure** or mechanical model by means of simplification and assumption, neglecting unimportant details and highlighting the essence and basic characteristics. Generally speaking, in structural analysis, the real structure is first simplified as a mechanical model, then the mathematical model, numerical model, and computer program are established, and finally the calculated results are applied to engineering analysis, design, and construction. This process is shown in figure 1.3, where the establishment and resolution of the associated

equations are two key links. Model verification and validation, which are closely related to structural analysis, are important research topics in the academic community. It is required that the theoretical methods of structural analysis should be verified by physical experiments and engineering practices.

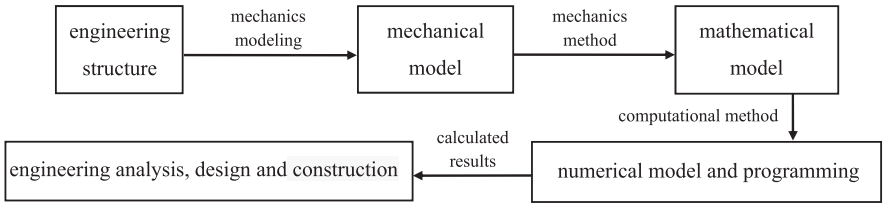


FIG. 1.3 – Process of computational modelling for structure.

The establishment of a computational model is the basis of mechanical computation and is extremely important. The established principles of structural computational model are presented as follows:

(1) Consistent with the actual situation—the computational model should not only reflect the main force and deformation characteristics of the actual structure but also make the calculated results accurate and credible.
(2) Easy to calculate—distinguishing the primary and secondary aspects, omitting the details, ignoring secondary factors, and making the analysis and computing procedure convenient and simple.

For constructing a computational model, it is necessary to simplify the real structure in some aspects. A brief description of the key points of simplification to establish the computational models of member structures is given as follows.

1.2.1 Simplification of Structural Systems

The member structure systems can be divided into plane member structures and space member structures. The general structures are actually space structures, where the parts are interconnected to form a spatial entirety in order to withstand the loads that may occur in all directions. However, in most cases, the actual structures can be decomposed into plane structures by ignoring some secondary spatial constraints, so that the computing procedure can be simplified. This textbook focuses on the computational issues of plane structures. Of course, there are also some structures that are not suitable to be simplified as plane structures because of their significant spatial characteristics and effects.

1.2.2 Simplification of Members

The cross-sectional dimensions (width and thickness) of members are usually much smaller than the length, and the stress on the section can be determined according to

its internal force (bending moment, axial force, and shear force). Therefore, in the computational model, the bars are represented by their axes, the connecting zones between the bars are represented by joints, the length of the bar is represented by the distance between joints, and the action points of the loads are also indicated at the axes.

1.2.3 Simplification of Joints

The interconnection zones of the bars are called **joints**. Usually, the joints are simplified into two ideal types:

(1) **Pinned joint** The connected members cannot move with respect to each other at the joints but can rotate slightly against each other, *i.e.*, they can transmit forces rather than moments. This ideal situation is difficult to be realized in practical engineering. The joints of the timber truss are closer to the pinned joints (figure 1.4a, b).

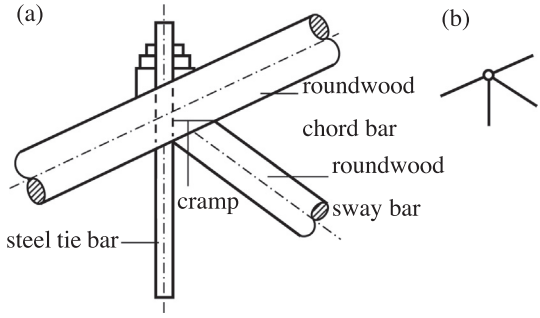


Fig. 1.4 – Pinned joint.

(2) **Rigid joint** The connected members can neither move nor rotate with respect to each other at the joints. They can transmit both forces and moments. The cast-in-situ reinforced concrete joint usually falls into this category (figure 1.5a, b).

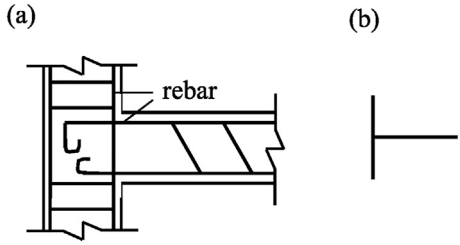


Fig. 1.5 – Rigid joint.

1.2.4 Simplification of the Supports

The connections between the structures and foundations are simplified as **supports**. According to mechanical characteristics, they are generally simplified into the following four types:

(1) **Roller support** The supported part can rotate and translate horizontally, and cannot move vertically (figure 1.6a). The reaction provided by roller support is only the vertical reaction force F_y. The roller support is represented by a vertical support link in the computational model (figure 1.6b). Roller bearings and rocker bearings used in bridge structures are actual examples of roller supports.

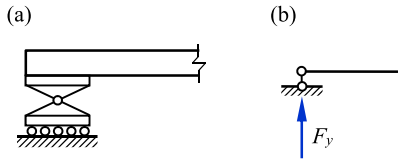


FIG. 1.6 – Roller support.

(2) **Pin support** The supported part can rotate and cannot translate (figure 1.7a). The pin support can provide two reaction forces F_x and F_y, and is represented by two intersecting support links in the computational model (figure 1.7b).

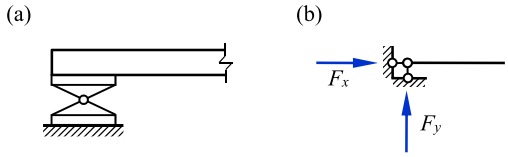


FIG. 1.7 – Pin support.

(3) **Slider support** The supported portion cannot rotate, but can slide in parallel along one direction (figure 1.8a). The slider support can provide a reaction moment M and a reaction force F_y and is denoted by two parallel support links in the computational model (figure 1.8b).

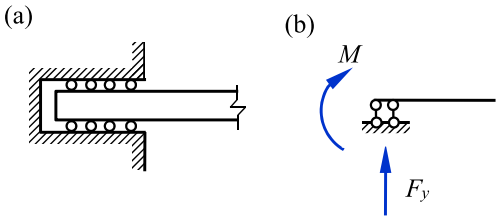


FIG. 1.8 – Slider support.

(4) **Fixed support** The supported part is completely fixed (figure 1.9a). The fixed support can provide three reactions F_x, F_y, M, and is represented in the computational model as figure 1.9b.

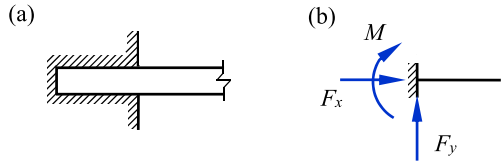


FIG. 1.9 – Fixed support.

1.2.5 Simplification of Material Properties

The materials used in the engineering structures are usually steel, concrete, brick, stone, timber, and so on. In structural computation, the materials for composing various members are generally assumed to be continuous, uniform, isotropic, elastic, or elastoplastic for simplification.

The above assumptions for metal materials with internal forces are in line with the actual situation in a certain range, whereas those for concrete, reinforced concrete, brick, stone, and other materials are approximations to a certain extent. As for timber, it is necessary to pay attention to the characteristic of anisotropy because its physical properties in the directions of grain and cross-grain are different.

1.2.6 Simplification of Loads

The loads applied on structures can be divided into body forces and surface forces. Body forces refer to the self-weight or inertia force of the structures, while surface forces are the acting forces transmitted to the structures by other objects through the contact surfaces, such as soil pressure, vehicle wheel pressure, and so on. The members are simplified as axes in member structures. Therefore, both the body forces and surface forces can be simplified as the loads acting on the axes of members. Moreover, the loads can also be simplified as concentrated loadings and distributed loadings in terms of their distributions.

To illustrate the simplification process of actual structures, an example of a single-story reinforced concrete industrial plant is exhibited in figure 1.10a. The beams and columns of the plant are prefabricated. The bottom ends of the columns are inserted into the cup rabbets of bases and then filled with fine aggregate concrete. The connections between the beam and column are realized by welding the embedded steel plates between the beam end and column top. Columns and beams form a framed bent in the transverse plane (figure 1.10b). The bent is connected by the roof panel and crane beam on the corbel.

The computational model shown in figure 1.10c can be applied to the mechanical analysis of the plant structure.

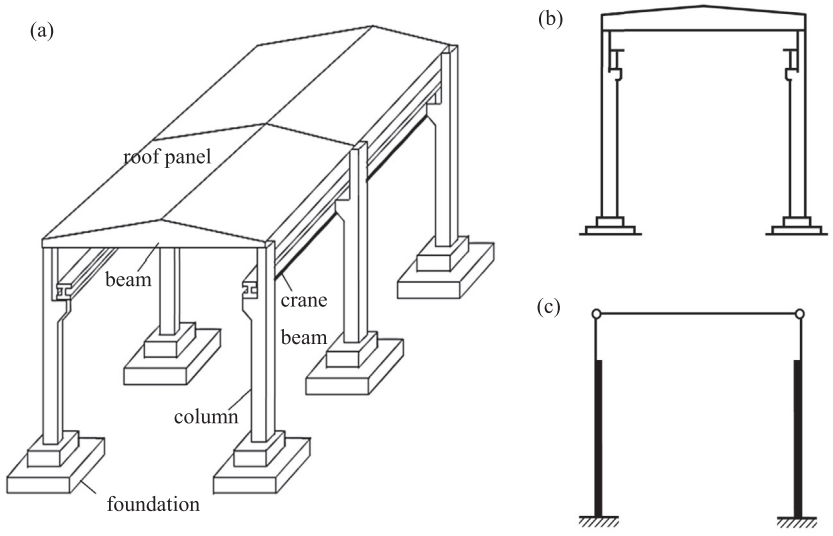


FIG. 1.10 – Simplification process of a single-story reinforcement concrete industrial plant.

First of all, although the plant structure is a space structure in which some framed bents are connected by roof panels and crane beams, the bents are regularly aligned with a certain interval in the longitudinal direction. Generally, the loads applied on the plant, such as dead load, snow load, and wind load, are evenly distributed along the longitudinal direction. Usually, these loads can be assigned to each of the bents, and each bent is regarded as an independent system. Hence, the actual spatial structure is simplified as a plane structure (figure 1.10b).

Then, the beams and columns are represented by their geometric axes. Since the sectional dimensions of beams and columns are much smaller than the length, the axes can be approximated as straight lines. Due to the difference in the sectional dimensions, the heavy line is used to indicate the thicker portion of the column.

The connections between beams and columns only depend on the welding of the embedded steel plates. Although the beam end and column top cannot translate relatively, it is possible that a slight relative happens. Hence, the connections between beams and columns in bent can be taken as the pinned joints. Because the relative translation and rotation cannot occur basically between the bottom of the column and the foundation, the bottom of the column can be viewed as fixed support.

1.3 Classification of Member Structures and Loads

1.3.1 Classification of Member Structures

The classification of structures actually refers to the classification of structural computational models. According to the mechanical characteristics, the member structures usually are divided into the following five categories:

(1) **Beam** The beam (figure 1.11a) is a flexural member whose axis is usually a straight line. The beam has a single-span or multi-span beam.
(2) **Arch** The axis of the arch (figure 1.11b) is a curve, and its mechanical characteristic is that there is a horizontal support reaction force (namely, thrust force) under vertical loads. Accordingly, the section of the arch has larger axial compression, smaller bending moment and shear force.
(3) **Truss** The truss (figure 1.11c) consists of straight bars, whose junctions are pinned joints. When the truss is subjected to joint loadings, the sections of its bars only yield axial force.
(4) **Frame** The frame (figure 1.11d) is also composed of straight bars, and the connections are usually rigid joints. The members of the frame are flexural members, and their internal forces include the bending moment, shear force, and axial force.
(5) **Composite structure** The composite structure (figure 1.11e) is a combination assembled by the truss and beam or frame, which contains composite joints.

The above-mentioned five types of structures are the most fundamental structural categories. In addition, there are the suspension structure and cable-stayed structure, where the main load-bearing members are the cables hanging on the towers and columns with only the tension.

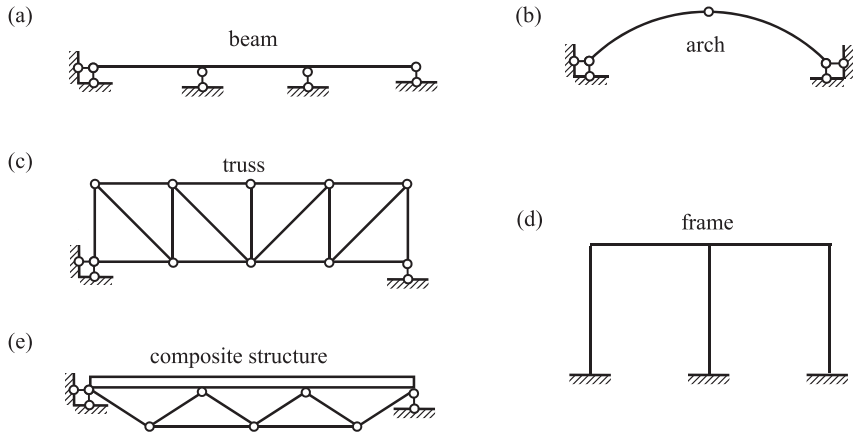


Fig. 1.11 – Various member structures.

According to the spatial position of the structural systems, the member structures are classified as plane structures and space structures. For plane structure, the axes of various members and action lines of external forces are all in the same plane, and figure 1.12 displays a plane truss structure. The space structure does not satisfy the above conditions. Figure 1.13 illustrates a space frame, where the axes of members are not in the same plane. In engineering design, most structures are usually viewed as plane structures for computation. In some cases, the spatial effect of structures must be considered.

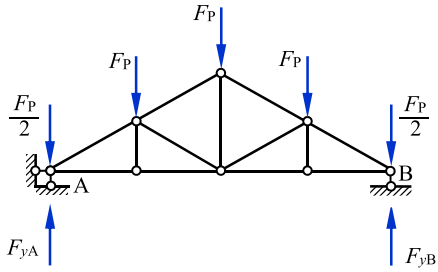


Fig. 1.12 – Plane truss.

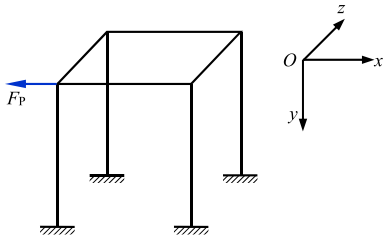


Fig. 1.13 – Space frame.

In addition to the above-mentioned classification, according to the computational characteristic, the structures are divided into statically determinate structures and statically indeterminate structures. The structures whose internal forces of members (including the support reactions) can be uniquely determined by the equilibrium condition are called **statically determinate structures**. If the internal forces of members and the support reaction forces cannot be uniquely determined by the equilibrium condition, and the deformation condition must be also considered, then the structures are called **statically indeterminate structures**.

1.3.2 Classification of Loads

Load is the external force actively applied to a structure, for example, the self-weight of the structure, the crane load in the industrial plant, the water pressure, and the

soil pressure acting on the structure. Besides the external force, other factors can cause internal force or deformation of the structure, such as temperature change, foundation settlement, material shrinkage, and so on. In a broad sense, these factors can also be called loads.

Before the calculation, the loading on the structure must be determined. The determination of load is an extremely important task in structural design. If the load is overestimated, the designed structure will be too bulky and wasteful. On the contrary, if the load is underestimated, the designed structure will not be safe enough. In terms of the action time, action position, and loading property, the loads can be classified into the following types.

(1) Classification according to the action time of the load

The loads can be divided into dead load and live load. Dead load is a constant load acting on the structure for a long time, such as the structural self-weight or soil pressure. Live load is a varying load temporarily acting on the structure, such as the moving train, crane load, crowd, snow load, and wind load.

(2) Classification according to the action position of the load

It is considered that the position of the dead load or most of the live load (such as snow load and wind load) acting on the structure is fixed, and such a load is named a **fixed load**. For a certain live load, *e.g.*, the load acting on a crane beam, or the vehicle load on a highway bridge, its position is mobile on the structure, which is called **moving load**.

(3) Classification according to the loading property

The loads are also divided into static load and dynamic load. The magnitude, direction, and position of **static load** do not vary or change very slowly with time, which does not cause significant acceleration of structure. Thus, the influence of inertial force can be neglected. The **dynamic load** is such a load that changes rapidly with time or suddenly acts or disappears in a short period, causing significant acceleration of the structure, and hence the effect of inertial force cannot be ignored. The self-weight and other dead loads of the structure are static loads. The load generated by the machine and the pressure of shock wave are examples of dynamic loads. Wind load and earthquake action are usually considered as dynamic loads, but sometimes they are simplified to static loads in structural design.

1.4 A Brief History of the Development of Structural Analysis

In the 17th century, several giants in the history of science laid the cornerstone of modern science—classical mechanics. In 1609 and 1619, J. Kepler ($1571 \sim 1630$) successively published *New Astronomy* and *Cosmic Harmony Theory* and proposed Kepler's three laws of planetary motion. In 1632, G. Galileo ($1564 \sim 1642$) published *Dialogues on Two World Systems of Ptolemy and Copernicus*, which supported

heliocentrism and firstly clarified the principle of relativity of the movement. In 1638, he published *Dialogues on Two New Sciences*, which discussed the problems of fracture resistance of materials, the resistance of medium to motion, the principle of inertia, the motion of free fall, the motion of an object on a slope, motion of a projectile, etc., and gave the definitions of uniform motion and uniform accelerated motion. In 1687, I. Newton (1643 ~ 1727) summed up the achievements of predecessors and published *Mathematical Principles of Nature Philosophy*, which expounded and established Newton's laws of motion and the law of universal gravitation, as well as constructed the basic framework of classical mechanics. Structural analysis was gradually developed under the framework of classical mechanics.

1.4.1 Energy Principles and Energy Methods

Energy principles are the theoretical foundation of structural analysis, from which several universal approaches of displacement calculation and internal force analysis, namely the energy methods, were derived. The principle of virtual displacements was proposed by J. Bernoulli (1667 ~ 1748) in 1717. British scientist J. C. Maxwell (1831 ~ 1879) established the theorem of reciprocal displacements in 1864 for a simple case with only two forces and proposed a unit load method for computing displacement. Subsequently, Italian scholar E. Betti (1823 ~ 1892) fulfilled a universal proof of Maxwell's theorem of reciprocal displacements in 1872 and generalized it to the theorem of reciprocal works. A. Castigliano (1847 ~ 1884) proposed Castigliano's first and second theorems and the principle of minimum work (*i.e.*, the principle of strain energy minimization or the principle of minimum potential energy) in 1879. German scholar F. Engesser (1848 ~ 1931) advanced the concept of complementary energy in 1884 and devised the method of complementary energy in 1889.

The principle of virtual work for deformable bodies means that external virtual work equals internal virtual work, which contains the principle of virtual forces and the principle of virtual displacements. From the principle of virtual forces for a deformable body, a general method for calculating the displacement or rotational angle of the specified point of beam, frame or truss can be derived, that is, the unit load method. As mentioned above, in 1864 Maxwell proposed the unit load method. In 1874, German mechanician O. Mohr (1835 ~ 1918) also independently established this method. Therefore, the unit load method is also called the Maxwell–Mohr method. Both of them had once applied the unit load method for computing the displacements of statically determinate trusses. The unit load method utilizes three basic conditions, namely, the equilibrium condition of the force, compatibility condition of deformation, and physical condition (or constitutive relation). Since the physical condition is not limited to describing linear elasticity, the unit load method is also suitable for solving the displacements of inelastic structures. Moreover, this method is applicable for both statically determinate and statically indeterminate structures. For any balanced rigid body system with ideal constraints, its internal virtual work is zero. Therefore, the virtual work done by the active forces (external

forces) of the rigid body system is zero, which is the virtual work principle of a rigid body. From the principle of virtual displacements for rigid body systems, an efficient and convenient method for calculating the specific constraint force of statically determinate structures was developed, that is, the unit displacement method.

In 1879, Italian railway engineer A. Castigliano published a monograph on the *Equilibrium Theory of Elastic System*, and based on the principle of virtual work, he advanced Castigliano's first and second theorems and the principle of minimum work. Castigliano's first theorem leads to the approach for determining the constraint force of a section after the displacements of the structure occur. This approach is appropriate to solve the forces of statically indeterminate structures. Its application is independent of the superposition principle and can be extended to nonlinear elastic systems, and it is equivalent to the force equilibrium condition of the structure. Castigliano's second theorem leads to the approach to solving the displacement of a point in trusses, beams, or frames. It can be extended to the nonlinear elastic structures and is equivalent to the compatibility condition of deformation. According to Castigliano's second theorem, the displacement at a point is equal to the first-order partial derivative of strain energy with respect to the force acting at this point along the direction of displacement to be solved. Castigliano's second theorem is used to determine the displacement of structures at a certain position on the premise that the load needs to apply on this position corresponding to the displacement to be solved. If there is no load corresponding to the displacement to be computed, the method of additional force can be employed. In 1878, F. Crotti proposed the theorem of calculating the displacement of elastic body. In 1889, F. Engesser also independently devised this theorem, which is called the Crotti–Engesser theorem. This theorem is derived based on the principle of virtual forces and the concept of strain complementary energy, and it is applicable to the displacement calculation of nonlinear elastic structures. Castigliano's second theorem is a special case of the Crotti–Engesser theorem. Hence, the principle of minimum potential energy and the principle of minimum complementary energy were established successively.

In 1886, German mechanician H. Muller-Breslau ($1851 \sim 1925$) developed a method for quickly determining the shapes of influence lines of internal forces of statically determinate and statically indeterminate beams based on the principle of virtual displacements. Namely, the influence line of a certain quantity (force) of interest is the same as the deflection shape of the released beam under this force. This method is called the Muller-Breslau principle or Muller-Breslau criterion, which is called the kinematic method in textbooks. In 2015, Dixiong Yang *et al.* established a kinematic method for analytically, exactly, and conveniently calculating the influence line equations of internal forces for statically indeterminate beam structures by utilizing the principle of virtual displacements for the deformable body, which overcomes the drawback of cumbersome computation of the past conventional approach for constructing influence lines of statically indeterminate forces.

In the middle of the 20th century, breakthrough progress had been made in the aspects of energy principles and energy methods. The researchers successively proposed the generalized variational principles with the two and three classes of

variables, which are different from the univariate variational principle (*i.e.*, the principle of potential energy and the principle of complementary energy), and provide a new approach for structural analysis. In 1950, E. H. Reissener advanced the generalized variational principle with two classes of variables in elasticity theory, showing the prospect of establishing equations that simultaneously satisfy different mechanical properties in the energy method. In the same year, Lingxi Qian ($1916 \sim 2009$) published a paper on the *Principle of complementary energy* in *Science China*, which pioneered the study of variational principles in the Chinese community of mechanics. Subsequently, some researches with international influence on variational principles appeared in China, a few of which were inspired by Lingxi Qian's paper. For instance, the generalized variational principle with three classes of variables was proposed by Haichang Hu ($1928 \sim 2011$) in 1954, and similar work was also published by K. Washizu in the United States in 1955, which was called Hu–Washizu variational principle.

1.4.2 Force Method and Displacement Method

Before the development of the force method and displacement method, one can only calculate the internal forces and displacements of simple structures. In the several decades since 1847, graphic methods (Maxwell diagram, etc.) and analytical methods (method of joints and method of sections) were applied to study the force analysis of statically determinate truss structures, and the foundation of truss theory was established. The force method was first proposed by Maxwell in 1864, then amended by Mohr in 1874, and fundamentally developed by Muller-Breslau in 1886. The basis of the force method is the compatibility condition of deformation, thus it is a compatibility method or flexibility method in essence. When the internal forces of statically indeterminate structures are solved by the force method, the redundant constraint forces are taken as the primary unknowns. Firstly, the geometrically unchangeable primary system is selected, and the canonical equations of the force method are established by the condition of deformation compatibility. Then, the influence coefficients of displacement in the equations of force method are obtained by the unit load method. Finally, the equations are solved to get the redundant forces. When the redundant forces are determined, the other support reactions and internal forces of statically indeterminate structures can be calculated by the static equilibrium condition.

The basic principle of the force method is simple and easy to understand. Nevertheless, for the complex statically indeterminate structures, the selection of the primary system requires more experience and intervention of users. Moreover, for the multistory multi-span frame structures, the number of redundant unknown forces is greater, and the difficulty of solving equations increases. The displacement method can overcome these difficulties. The development of the displacement method is divided into two stages of the slope-deflection method and the matrix displacement method. In 1826, the French mechanician C. L. M. H. Navier ($1785 \sim 1836$) advanced the displacement method in elasticity mechanics and applied it to solve the internal forces of statically indeterminate trusses.

The secondary bending moment method as the predecessor of the slope-deflection method was first proposed by the German H. Manderla in 1880 for computing the secondary bending stress of the truss, and it was modified by Mohr in 1892 and gradually became known. The secondary bending moment method assumes that the moment does not affect the joint displacements of truss structures, which is only effective for small bending moment effects. In the 20th century, with the gradual increase of bridges and high-rise buildings built with reinforced concrete material, the bending moment effect caused by the rigid joints became significant, and the joint displacements yielded by the axial forces were small. In 1914, Danish engineer A. Bendixen applied the slope-deflection method for calculating the internal forces of the frames with sidesway. In 1915, American scholars W. M. Wilson and G. A. Maney modified the secondary bending moment method and independently used it to solve the internal forces of frames, and called it the slope-deflection method. In 1926, Danish scholar A. Ostenfeld pointed out the duality between the force method and the displacement method. In fact, the canonical equations with respect to unknown displacements in the structures are complementary to the canonical equations for redundant forces. When using the displacement method to solve the internal forces of the structure, the independent rotational angles (slopes) and linear displacements (deflections) are taken as the primary unknowns, and the whole structure is divided into a number of single members. Based on the force–displacement relation (the slope-deflection equation) of the members, the canonical equations of the displacement method are constructed by the equilibrium condition of joint moments and section projection, and then the displacements of joints are obtained by solving the equations. Finally, the displacements of joints are back substituted into the slope-deflection equations of members to obtain the internal forces of member ends. The displacement method does not need to care whether the primary system is geometrically unchangeable or not (note that the primary system of the force method cannot be a geometrically changeable system), and it is suitable for the mechanical analysis of statically determinate and statically indeterminate structures.

It is seen that both the force method and the displacement method utilize the static equilibrium condition, the deformation compatibility condition, and the force–displacement relation (which can be analogous to the stress–strain relation of differential elements) of the structure. The difference lies in that for the force method, the force–displacement relation is constructed firstly by the unit load method at the level of the primary system (the flexibility influence coefficients representing the displacements of the primary system under the unit load), and then the deformation compatibility condition is satisfied and the flexibility equations of the force method are established at the structural level. Finally, the internal forces of member ends are solved *via* the equilibrium condition. While for the displacement method, firstly the deformation compatibility condition is considered at the level of the member element to express the force–displacement relation (the slope-deflection equation or the stiffness influence coefficients representing the internal forces of the primary structure under the unit displacement). Subsequently, the static equilibrium condition is met at the structural level to form the stiffness equations of the displacement method, and the obtained joint displacements are used to solve the

internal forces of member ends according to the deformation compatibility condition and force–displacement relation.

It is interesting to note that, there exist close connections between the force method, the displacement method, and the energy principles. The equation of the force method can be derived from the principle of complementary energy or Castigliano's second theorem, and the equation of the displacement method can be deduced from the principle of potential energy. Moreover, the development of the finite element method has a direct and profound connection with the variational principle. In general, the displacement method is suitable for solving statically indeterminate and determinate structures, while the force method is generally applicable to indeterminate structures. The displacement method can directly solve structural displacements and internal forces, while the force method cannot straightforwardly determine displacements. Additionally, the displacement method is easier to establish the stiffness equations and is convenient for the computer to solve automatically. Therefore, the displacement method is the key method of structural analysis. It is worth pointing out that the force method introduced in the textbook takes redundant constraint forces as unknowns, while some efforts are also being made to develop the integrated force method, which is parallel to the displacement method and takes the whole forces as unknowns.

Before the advent of the computer and matrix displacement method, it was very troublesome for engineers to solve the higher-order algebraic equations obtained by the slope-deflection method. In 1922, Calisev proposed a successive approximation method by taking the rotational angles of the frames without sidesways as unknowns, which can avoid directly solving the higher-order equations. In 1930, American scholar H. Cross developed a successive displacement method named the moment-distribution method, to approximately solve the statically indeterminate continuous beams. In fact, the ideas of the successive approximation method and the moment-distribution method are almost the same. The calculation process of the moment-distribution method is to constrain and relax the joints one by one, distribute and balance the bending moments at the joints in turn, and repeat the iteration until the unbalanced bending moments or the constraint moments of the joints are close to zero. The moment-distribution method applies successive iterative strategies to solve the simultaneous algebraic equations of the displacement method, which overcomes the difficulty of directly solving higher-order equations. It was a successful application of the iterative method for solving linear algebraic equations in structural mechanics, which attracted great attention in the engineering community and was the most significant progress in structural analysis in the 1930s. However, after the invention of the computer, the moment-distribution method and other successive analysis methods, such as the moment-iteration method, the no-shear distribution method, and the Southwell relaxation method, gradually retreated from the center of the historical stage and gave way to the matrix displacement method which emerged in the 1950s.

1.4.3 Matrix Displacement Method and Finite Element Method

Firstly, the development of the finite element method is introduced, and then a specific description is given. The idea of the finite element method had been explicitly proposed by mathematician R. Courant (1888 ~ 1972) in 1943. In the 1950s, J. H. Argyris (1913 ~ 2004) and R. W. Clough (1920 ~ 2016) *et al.* developed the finite element method from aircraft structural analysis. In 1960, Clough coined the term "finite element". Many researchers such as Chinese mathematician Kang Feng (1920 ~ 1993), British mechanician O. C. Zienkiewicz (1921 ~ 2009), and American scholar Xuehuang Bian (1919 ~ 2009) had made important contributions to the development of the finite element method.

In 1943, Courant first applied the piecewise interpolation function defined on the triangle region and solved the Saint–Venant torsion problem in conjunction with the principle of minimum potential energy. Zienkiewicz wrote in a review article: "It is a pity that, not being an engineer, he did not link the idea with networks of discrete elements and that, for this reason, I believe, his work had to lie in obscurity for so many years." Later, the great significance of Courant's work was recognized, and the year 1943 was regarded as the birth year of the finite element method.

After the 1940s, aeronautical engineers needed to perform detailed stress analysis of continua such as plates, frames, beams, and trusses in metal aircraft structures, and the finite element method originated from the structural analysis of these continua. Greek mechanician Argyris published a series of papers in the journal *Aircraft Engineering* from October 1954 to May 1955. They were reprinted as a monograph entitled *Energy Theorems and Structural Analysis* in 1960. This monograph contains the following two main contributions. Firstly, Argysis generalized and unified the basic energy principles of elastic structure analysis. Secondly, he developed a practical method for complex structural analysis of aeronautical engineering, namely, the matrix analysis method, including the matrix displacement method and matrix force method. The matrix displacement method establishes the elemental stiffness matrices by means of the slope-deflection equation, and assembles them into the global stiffness matrix to form the global stiffness equation of the displacement method, then solves the equation to obtain the displacements of nodes, and further achieves the internal forces of members through the back substitution into the elemental stiffness equation. The matrix displacement method is the prototype of the finite element method and can be derived from the principle of minimum potential energy. It can also be called the finite element method of member structures. The attributes of the matrix displacement method lie in that, the establishment of the primary system is simple and uniform, the formation of elemental and global stiffness matrices is modular, and the solution of equations is easy to program. Accordingly, the matrix displacement method fairly conforms to the requirement of computer automatic calculation and is widely applied owing to its generality. The matrix force method was no longer used because it inherited the disadvantages of the force method requiring the selection of a primary system and possessed the limitation that it is difficult to calculate automatically. In 1965, the *Theory of Structures* (2nd Edition) co-authored by S. P. Timoshenko (1878 ~ 1972)

and Young introduced the matrix force method and the matrix displacement method.

In the summer of 1952, Clough came to Boeing company to work on the structural analysis of the aircraft wing under the guidance of M. J. Turner and attempted to calculate the influence coefficients of the box beam with a small aspect ratio, but the displacement value obtained by Levy method differed from the measured value up to $13\% \sim 65\%$. In the summer of the following year, Clough returned to Boeing, and Turner suggested a better way to model the skin panel. Clough proposed the Ritz analysis method in combination with the simple strain field. In 1956, Turner and Clough *et al.* published a famous paper on the analysis of two-dimensional elastic structures, in which they utilized three types of elements, namely triangle, quadrilateral and rectangular elements, to discretize the continuum. Actually, they advanced the finite element method which adopted the idea of the direct stiffness method (matrix displacement method) for the analysis of member structures, and the finite element used was the displacement element. The elemental stiffness matrices and the structural global stiffness equation were established in that paper. This work is considered to be the beginning of the finite element method in the engineering community. In 1960, Clough first named the finite element and opened up a broad way to transform the continuum mechanics problems into discrete mechanical models. E. L. Wilson, as Clough's early Ph.D. student, made an important contribution to computer simulation based on the general program of structural analysis and developed the currently popular SAP software system.

In the early 1960s, Kang Feng independently and separately developed the mathematical theory of the finite element method to address the stress analysis problem of the dam. In 1964, he established the finite element method for numerically solving the elliptic partial differential equations, which formed a standard algorithm, and compiled a general computer program for engineering structure analysis. In 1965, an article on *Difference schemes based on variational principles* was published. The properties of convergence and stability of the method were proved under broad conditions. Kang Feng is a computational mathematician with an international reputation. He also developed the natural boundary element method and proposed the symplectic geometric algorithm for solving the Hamilton system, which opened a new way for the long-term structure-preserving computation of dynamic systems.

From 1963 to 1964, the study of J. F. Besseling, R. J. Melosh, and R. E. Jones indicated that the foundation of the finite element method (still called the direct stiffness method in the paper) is the variational principle. They proved that the finite element method is another formalism of the Ritz analysis method based on the variational principle, and confirmed that it is a universal method for dealing with continuum problems. Their research built the bridge of communication between the mathematics and engineering circles for a better understanding of the finite element method and pushed people to realize that it is a numerical method with both a rigorous theoretical basis and a common application value, and thus promoted the development of the finite element method.

In 1967, O. C. Zienkiewicz and Y. K. Cheung published the first book about the finite element method in the world, *The Finite Element Method in Structural Mechanics*, and later adapted and published the three-volume book *The Finite*

Element Method with R. L. Taylor. The seventh edition of this book was published in 2013, which made a great contribution to the development and dissemination of the finite element method. Zienkiewicz was engaged in the research of finite element method for a long time, and his outstanding contributions include that: he proposed the patch test approach and the isoparametric element, developed the error estimation method, as well as promoting the application of the finite element method in fluid mechanics and civil engineering.

To sum up, the finite element method was a novel numerical method for solving problems in continuum mechanics, physics, and various kinds of engineering sciences developed in the 1950s and 1960s. Specifically, the infinite degrees of freedom of the continuum are discretized into finite degrees of freedom through the subdivision of finite elements, so that the problem is reduced to solving algebraic equations based on variational principles or other methods. The finite element method not only has the advantages that its theory is complete and reliable, its formulation is simple and standard, and its precision and convergence are guaranteed, but also can construct appropriate elements according to the nature of the problem. Hence, the finite element method has a wider application range than other numerical approaches. The marriage of the finite element method and the computer has produced tremendous power, and it has become an indispensable tool for scientific research and technology development in many fields. Its application range is quickly extended from simple bar, beam, and plate structures to complex spatial composite structures. The finite element method makes the static analysis of some large-scale complex structures impossible in the past to become routine calculation, and numerous dynamic problems and nonlinear problems also have corresponding solution schemes. For the mechanicians, with the help of the finite element method, credible numerical results of many problems difficult to obtain analytic solutions can be achieved. For engineering and technology professionals, the design of many complex engineering systems can not rely on or less depend upon expensive experiments. At present, the finite element method has become a crucial method to solve mathematical physics equations and is popularly used in computational mathematics, computational physics, computational mechanics, electromagnetics, heat transfer, meteorology, geophysics, engineering science, and other fields. Indeed, the finite element method has truly revolutionized current engineering and scientific environments.

From the magnificent development history of structural analysis, we can not only understand the evolution process of mechanical theories and methods driven by the demand of engineering application, but also can appreciate the beauty of mechanics and the power of scientific innovation, and thus inspire the passion of scientific discovery and creation.

1.5 A Brief Introduction to Important Figures in Structural Analysis

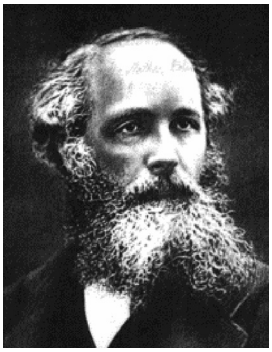
In the long history of the development of structural analysis, many important figures had emerged, who had made outstanding scientific contributions. The following

introduction highlights six great figures: C. L. M. H. Navier, J. C. Maxwell, O. Mohr, H. Müller-Breslau, Stephen P. Timoshenko, J. H. Argyris.



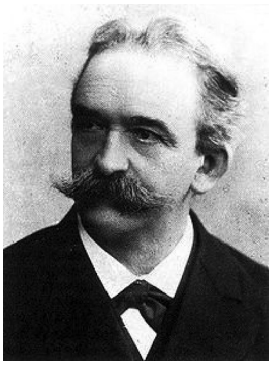
C. L. M. H. Navier (1785∼1836), was a French mechanician and engineer. In 1802, he entered the Paris Polytechnic School for study. Then, he entered the Bridges and Roads School in 1804 and graduated in 1806. He taught applied mechanics at the Bridges and Roads School since 1818 and was a professor from 1831. In 1824, he was elected as a member of the French Academy of Sciences.

Navier's main contribution lies in that he established the basic equations for fluid mechanics and elasticity mechanics, respectively. In 1821, he generalized Euler's equation of fluid motion, took into account the forces between the molecules, and thus constructed the basic equations of fluid equilibrium and motion, which contain only one viscous constant. In 1845, Stokes improved his motion equation of fuild dynamics from the continuum model and obtained the form of rectangular coordinate components of fluid motion equation (later called the Navier–Stokes equation) with two viscous constants. In 1821, Navier also derived the equilibrium and motion equations of elastic solids from the molecular model (published in 1827). These equations contain only one elastic constant. The basic equations of isotropic elastic mechanics with two elastic constants were derived by Cauchy in 1823. In 1826, he proposed the idea of the displacement method in elasticity mechanics and used it to solve the internal force of a statically indeterminate truss. In addition, Navier's other achievements in mechanics include the following: he first (in 1820) utilized double trigonometric series to solve the fourth-order partial differential equation of simply supported rectangular plate and introduced mechanical work in engineering to measure the efficiency of the machine. In civil engineering, he changed the tradition of designing suspension bridges by only relying on experience, and adopted theoretical calculations in structural design.



James Clerk Maxwell (1831 ∼ 1879), was a British physicist. In 1854, he graduated from Trinity College, Cambridge University, and earned a degree in mathematics. In 1856, he became a professor of natural philosophy at Marischal College in Aberdeen, Scotland. In 1860, he was a professor of natural philosophy and astronomy at London King's College. In 1861, he was elected as a member of the Royal Society of London. In the spring of 1865, he resigned from his faculty and returned to his hometown to systematically summarize the research results in electromagnetics and complete the classical great work *Treatise on Electricity and Magnetism* about electromagnetic field theory, which was published in 1873. In 1871, he was appointed as the newly established Cavendish experimental physics professor at Cambridge University and was responsible for the preparation to establish the famous Cavendish Laboratory, which was built in 1874 and he served as the first director until his death in 1879.

The most significant achievement of Maxwell is that he laid the foundation of classical electromagnetic theory. He also made important contributions to other fields of physics, such as thermodynamics, statistical physics, molecular physics, and optics. Maxwell's contribution to mechanics mainly includes the following aspects: in 1853, he generalized the method of measuring stress by polarized light; in 1864, he established the theorem of reciprocal displacements, and proposed the graphical method for solving the internal forces of trusses in structural analysis; he pointed out that the truss shape figure and the internal force figure are reciprocal, advanced the unit load method (also called Maxwell–Mohr method) for calculating the displacements of statically indeterminate trusses, and also developed the force method for analysis of indeterminate structures at the same time. In 1868, he proposed a model for characterizing viscoelastic materials (later referred to as the Maxwell model) and conceived the concept of relaxation time. In the same year, in his article *On governors*, the problem of motion stability of automatic governors of steam engines and clock mechanisms was investigated. In 1870, Maxwell extended the stress function in elasticity theory proposed by G. R. Airy from a 2-dimension to a 3-dimension case and pointed out that it should satisfy the biharmonic equation. In 1873, he gave the stress field caused by attractive and repulsive force in the electrical charged system.



Christian Otto Mohr (1835 ∼ 1918), was a German mechanician and civil engineer. In 1851, he studied at the Hannover Institute of Technology and worked in the railway sector in Hannover and Oldenburg since 1855. He designed some famous bridges and promoted the early application of steel truss structures. In 1867, he became a professor of mechanics at the Stuttgart Institute of Technology, and in 1873 he taught at the Dresden Institute of Technology. His lectures were straightforward and unpretentious, and he was popular with the students.

Mohr had made great achievements in the fields of structural analysis and mechanics of material. In 1874, he independently developed the Maxwell–Mohr method of structural analysis, which can solve the nodal displacements of the trusses, also known as the virtual force method or the unit load method. The Maxwell–Mohr method can also determine the internal forces of the bars of statically indeterminate trusses, and it is a force method in this situation. Mohr was keen on developing the graphical method of mechanical analysis. In 1882, based on the Carl Culmann stress circle, he proposed the Mohr circle characterizing the 2-dimensional and 3-dimensional stress states, and thus established the shear stress-based strength theory. He also constructed the Williot–Mohr diagram for solving the truss displacements. In addition, Mohr and Emil Winkler independently suggested the concept of the influence line.



Heinrich Müller-Breslau (1851 ∼ 1925), German mechanician and engineer. He was born in Breslau, Germany in 1851, and his surname was changed by himself to

avoid resemblance to others. After graduating from the Realschule in 1869, he became an engineer of the Prussian army and participated in the Franco-Prussian War from 1870 ~ 1871. Later, he studied civil engineering and mathematics at the Berlin School of Architecture and the University of Berlin. From 1875, he worked as an independent consulting engineer mainly working on the design of iron bridges. In 1885, he became a professor of civil engineering at the Hannover Institute of Technology. A series of published papers and books as well as some famous engineering designs quickly established his good reputation. In 1888, he succeeded Emil Winkler as the structural engineering director of the Berlin Institute of Technology and later became the dean.

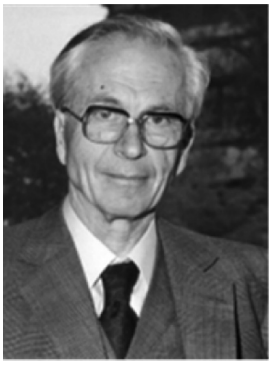
Müller-Breslau is considered to be the founder of modern structural engineering in Germany and made important contributions to the theory of beam and frame analysis in structural analysis. His numerous works refined and elaborated the existing theoretical methods and expressed them systematically and uniformly, and had a profound impact. He developed fundamentally the force method for analysis of statically indeterminate structures and proposed the kinematic method for drawing the influence lines of internal forces of statically determinate structures under moving load, and resolved many problems in structural theory, including the analysis of cantilever beam, arch structure, and lattice structure, soil pressure problems of retaining walls, and stability issue of straight members. His three volumes of handbook *Graphical Statics of Structural Theory* was the authoritative work about graphic mechanics of graphic mechanics which achieved international recognization. The major projects designed by Müller-Breslau included the Volga River bridge in Kazan, Russia, and the new cathedral in Berlin. He also participated in the manufacturing of airships, designed large-scale aircraft hangars, and introduced cantilevered wings to aircraft design.



Stephen P. Timoshenko (1878 ~ 1972), Ukrainian American engineering mechanician, and founder of modern engineering mechanics. He was born in Ukraine in 1878 and graduated from the Institute of Engineers of Ways of Communication in St. Petersburg in 1901. He worked in the Institute of Engineers of Ways of Communication, St. Petersburg Polytechnic Institute, Kiev Institute of Technology, Zagreb Institute of Technology, Philadelphia Vibration Company in the USA,

Westinghouse Electric Company, University of Michigan, and Stanford University. He was engaged in the study of structural analysis, mechanics of materials, and elasticity theory, made great groundbreaking contributions in many aspects, and was known as the "father of modern engineering mechanics". In order to remember his great achievements in engineering science and education, the American Society of Mechanical Engineers established the Timoshenko Medal in 1957, and he was the first recipient. He received numerous honors and awards, and was elected as a member of the seven National Academy of Sciences: Ukrainian Academy of Sciences (1918), Russian Academy of Sciences (1928), Polish Academy of Sciences (1935), French Academy of Sciences (1939), National Academy of Sciences of USA (1941) The Royal Society of London (1944), Italian Academy of Sciences (1948).

Some works of Timoshenko in structural analysis, mechanics of materials, and elasticity theory were written into standard textbooks and had become the fundamental knowledge in the field of engineering mechanics. In particular, he developed the energy method of structural stability analysis and established the theory of transverse vibration of deep beams, namely the theory of the Timoshenko beam. Also, he proposed the concept of shearing center of beam and the theory of warping and torsion of thin-walled members, and promoted the development of elasticity mechanics. He published 12 textbooks, including *Strength of Materials*, *Elements of Strength of Materials*, *Theory of Elasticity*, *Theory of Plates and Shells*, *Theory of Elastic Stability*, *Theory of Structures*, *Vibration Problems in Engineering*, *Advanced Dynamics*, and *History of Strength of Materials*. These classical books of mechanics have been translated into 35 languages and distributed around the world, and are still in use today. These books are characterized by conciseness and clarity, describing complex phenomena and problems in straightaway language, often starting with simple and specific problems and generalizing them into more complex situations. In fact, his teaching style is similar, which is deeply beloved and welcomed by many students.



John Hadji Argyris (1913 ∼ 2004), Greek mechanician. He was born in Volos, Greece in 1913. After studying civil engineering at the National Technical University of Athens, he obtained his Engineering Diploma at the Technical University of Munich in 1936. He spent his early career in the design and research of complex

structures and continued his studies in aeronautical engineering at the Swiss Federal Institute of Technology (ETH Zurich) from 1941 to 1942. In 1943, he joined the Engineering Department of the Royal Aeronautical Society of London. Since 1949, he was a senior lecturer in aeronautical engineering at the Imperial College of the University of London. In 1955, he was promoted to professor and director. In 1959, he became the director of the Institute for Statics and Dynamics of Aerospace Structures at the University of Stuttgart, Germany. Together with Ray W. Clough and Olgierd C. Zienkiewicz, Argyris was the leading figure in developing the finite element method following Richard Courant. He proposed the matrix force method and matrix displacement method for structural analysis, made far-reaching and fundamental contributions to computational mechanics and its industrial applications, and promoted the research of chaos theory. He was the author of: *Energy Theorems of Structural Analysis, Introduction into the Finite Element Method, Dynamics of Structures, An Exploration of Chaos.*

He was elected as the Fellow of the Royal Society of London (1986), and a Fellow of the Royal Academy of Engineering (1990). In addition, he received many awards, including the Gauss–Newton Award from the International Association for Computational Mechanics (1990), the von Karman Award from the American Society of Civil Engineers (1975), the Timoshenko Medal from the American Society of Mechanical Engineers (1981), and the Royal Medal of the Royal Society (1985), the Prince Philip Gold Medal of Royal Academy of Engineering (1996), the Grand Cross of Merit of the Federal Republic of Germany (1989), the Einstein Award from the Einstein Foundation, USA (2000).