A Text Book of

MATHEMATICS FOR MACHINE LEARNING

(COURSE CODE 314320)

For

Semester - IV
Second Year Diploma in
Artificial Intelligence / Artificial Intelligence and Machine Learning / Data Science (AI/AN/DS)

As Per MSBTE's 'K' Scheme Syllabus w.e.f Academic Year 2024-2025

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Preface ...

We feel great pleasure in presenting this book on **Mathematics for Machine Learning** for Semester - IV of Diploma Engineering Covering the latest syllabus prescribed by **MSBTE for 'K'-scheme** w.e.f. academic year 2024 - 2025.

The book consists of six chapters. Each chapter is in accordance to the prescribed syllabus.

Keeping in mind the time limit of the semester and the difficulty level of students, We have included large number of varied solved problems. This will help students to know the trend of the questions asked in real examination. To enhance their learning, every sub-topic is provided with logically arranged exercises. It is our humble request to students that they should solve each and every exercise at least once. This will not only boost your interest in Mathematics but also enhance confidence in your work methodology.

If you wish to get high score on Mathematics, merely buying this book will not help. You have to work hard, smart and regular.

Remember, there is no magic formula to become virtuoso overnight. You have to work diligently. "There are no short-cuts to success."

All valuable suggestions for the enrichment of the book will be highly well-comed and gratefully acknowledged.

Special thanks go to Shri Dinesh Furia, Shri Pradeep Furia and Shri Jignesh Furia of M/s Nirali Prakashan for their willingness and co-operation in high standard production of a book.

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Lastly but not least we are very much thankful to Mr. Ilyas Shaikh for their excellent computer typesetting and Ms. Yojana Deshpande for Figures Drawing.

We will deeply appreciate your responses in relation to any printing error or any innovative suggestions to improve the quality of the presentation in this book.

Authors

ABOUT THIS BOOK

This book is designed specifically for two purposes :

(1) To cover up the entire latest syllabus prescribed by MSBTE for 'K'-Scheme.

(2) To boost students interest in the Mathematics through careful selection of problems relevant to their mental level of understanding.

Although the mathematics is difficult, it is at the same time very learnable. There is no magic to master it overnight.

Mathematics can be mastered through hard work, analytical thought and by training yourself to think unlike the Author. Exercises are designed to prompt you to think like the Author.

Practice illustrative problems first by writing steps of solution using pen and paper. Don't just read this book – study it, scrutinize it! In short, for the next four months, this book should be a part of your life.

Knowing how the problems are solved and the authors think in the process of solving problems will give you insight into the subject and significantly enhance your confidence in mathematics. The more you practice, the more you understand it. This will enable you enjoy every examination taken by college or board or any agency.

Remember!

- Only way to boost confidence is practice! practice!! and practice!!!
- Each chapter has been carefully designed followed by practice exercises to keep your interest alive.
- Every illustrative example is self-explanatory.
- In each chapter, past years MSBTE Question Papers Problems are provided to make you aware of the nature of problems asked in MSBTE examinations.
- Regular use of this book will reinforce you learning skills, critical thinking skills and organizational skills etc.

We hope you will enjoy this book the way we enjoyed making it.

Good Luck!

Authors

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1.2 Partial Derivative (Two variables): Introduction, partial derivative of first order, second order and mixed order.

1.3 Homogeneous Function.

1.4 Euler's theorem on homogeneous function (Two variables).

1.5 Maxima and minima of function (Two variables).

1.6 Lagrange's method of undetermined multipliers with one constant (Two variables).

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3.3 Algebra of vectors: Addition of vectors, Triangle law of vectors addition, Parallelogram law of vectors addition, Subtraction of vectors, Multiplication of vectors by scalar.

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Unit - V Linear Programming Problems

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Chapter 1...

Partial Differentiation

Weightage of Marks = 14, Teaching Hours = 10

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1.2 Partial Derivative (Two variables): Introduction, partial derivative of first order, second order and mixed order.
1.3 Homogeneous Function.
1.4 Euler's theorem on homogeneous function (Two variables).
1.5 Maxima and minima of function (Two variables).
1.6 Lagrange's method of undetermined multipliers with one constant (Two variables).

Unit Outcomes

After reading this chapter, students will be able to

- Calculate partial derivative of first order, second order and mixed order.
- Calculate maximum and minimum value of the function.
- Verify Euler's theorem for the homogeneous function.
- Find maximum and minimum value of the function subject to given condition using Lagrange's undermined multipliers.

1.1 INTRODUCTION TO DERIVATIVE AND INTEGRATION

Derivative and integrals are fundamental concepts in calculus, a branch of mathematics developed by Sir Isaac Newton and Gottfried Wilhelm Leibniz in the late 17th Century. These concepts are used to study how quantities change and to find areas under curves, respectively. Let us explore each of them in more detail.

Derivative : A derivative measures how a function changes as its input (usually denoted by x) changes. In other words, it represents the rate of change of a function at a particular point on it. The derivative of a function $y = f(x)$ is often denoted as $\frac{dy}{dx}$ (read as "dee y by dee x") or $f'(x)$.

Mathematically, the derivative of a function f (x) at a specific point x = a is defined as :

$$f'(x) = \lim_{h \to 0} = \frac{f(a+h) - f(a)}{h}$$

This is also known as first principle of derivatives. Using this, we can obtain derivatives of several basic functions. For example :

Let $y = f(x) = x^3$

$\therefore \quad f(x+h) = (x+h)^3$

Then, from first principle of derivatives,

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(x+h-x)\{(x+h)^2 + x(x+h) + x^2\}}{h} \quad \ldots \text{Using } a^3 - b^3 = (a-b)\cdot(a^2 + ab + b^2)$$

$$= \lim_{h \to 0} \frac{h \cdot \{(x+h)^2 + x(x+h) + x^2\}}{h} \quad \ldots h \neq 0$$

$$= \lim_{h \to 0} \{(x + h)^2 + x (x + h) + x^2\}$$
$$= \{(x + 0)^2 + x (x + 0) + x^2\}$$
$$= x^2 + x^2 + x^2$$
$$= 3x^2$$

Thus, $\frac{d}{dx}(x^3) = 3 \cdot x^{3-1} = 3x^2$

In the above expression, h represents a small change in the input x. As h approaches to zero, the quotient $\frac{f(x + h) - f(x)}{h}$ approaches the instantaneous rate of change of the function at any point on it. The derivative provides the slope of the tangent line to the graph of the function at that point.

Derivatives have many practical applications, such as calculating velocities, accelerations, optimization problems, and understanding the behaviour of functions.

The notation $\frac{dy}{dx}$ or $\frac{d}{dx}(y)$ or $\frac{d}{dx}[f(x)]$ represents the derivative of the function y = f (x) w.r.to x. It is also written by any one of the following notations.

y', $f'(x)$, y_1, $\dot{y}$.

The process of finding derivative of a continuous function is called **differentiation**.

List of Standard Derivatives :

We make a list of derivatives of **basic functions** as below for ready reference. Students are advised to know them with proper understanding. Without knowing them, you will be handicapped for future study. Do not neglect them. Try to memorize and use whenever required. The best way of study them is make a copy on hard paper and keep along with you while solving problems on derivatives.

(1) $\frac{d}{dx}(k) = 0,\ k \neq 0,\ k \in R$, is a constant

It follows that $\frac{d}{ds}(3) = 0,\ \frac{d}{dr}(\pi) = 0$

Thus, derivative of a constant w.r.to any variable is zero.

(2) $\frac{d}{dx}(x)$ or $\frac{dx}{dx} = 1$

It follows that $\frac{d}{dp}(p) = 1,\ \frac{d}{dr}(r) = 1$

Thus, derivative of any variable w.r.to itself is zero.

(3) $\frac{d}{dx}(kx) = k \frac{d}{dx}(x) = k(1) = k$, where k is a constant coefficient.

Note that while taking derivative of a function, the constant coefficient remains constant (as it is).

(4) $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$, n is rational, positive, negative, fraction.

(5) $\frac{d}{dx}(k \cdot x^n) = k \cdot n \cdot x^{n-1}$

(6) $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n}{x^{n+1}}$

(7) $\frac{d}{dx}\left(\frac{k}{x^n}\right) = \frac{-k \cdot n}{x^{n+1}}$

(8) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

(9) $\frac{d}{dx}\left(\frac{k}{x}\right) = \frac{-k}{x^2}$

(10) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

(11) $\frac{d}{dx}(e^x) = e^x$

(12) $\frac{d}{dx}(a^x) = a^x \cdot \log_e a, \quad a \neq 0, \quad a \neq 1, \quad a \in R, \quad a > 0$

(13) $\frac{d}{dx}[\log_e x] = \frac{1}{x}, \quad x > 0$

(14) $\frac{d}{dx}[\log_e |x|] = \frac{1}{x}, \quad x \neq 0$

(15) $\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{\log_e x}{\log_e a}\right]$ … By rule of change of base

$= \frac{1}{\log_e a} \cdot \frac{1}{x}$ … $\frac{1}{\log_e a}$ is treated as constant coefficient

(16) $\frac{d}{dx}[\sin x] = \cos x$

(17) $\frac{d}{dx}[\cos x] = -\sin x$

(18) $\frac{d}{dx}[\tan x] = \sec^2 x$

(19) $\frac{d}{dx}[\cot x] = -\operatorname{cosec}^2 x$

(20) $\frac{d}{dx}[\sec x] = \sec x \cdot \tan x$

(21) $\frac{d}{dx}[\operatorname{cosec} x] = -\operatorname{cosec} x \cdot \cot x$

Note that derivative of trigonometric functions beginning with latter 'c' are all negative.

(22) $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$

(23) $\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$

(24) $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$

(25) $\frac{d}{dx}[\cot^{-1} x] = \frac{-1}{1+x^2}$

(26) $\frac{d}{dx}[\sec^{-1} x] = \frac{1}{x \cdot \sqrt{x^2-1}}$

(27) $\frac{d}{dx}[\operatorname{cosec}^{-1} x] = \frac{-1}{x \cdot \sqrt{x^2-1}}$.

Observe derivatives of inverse trigonometric functions from (22) through (27) in pairs. They are equal except those starting with latter 'c' are negative.

Rules of Derivatives :

We shall establish working rules for finding derivatives of composed functions.

Derivative of Sum or Difference of Functions

Statement : If $y = u \pm v$, where u and v are differentiable functions of x, then,

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

Derivative of Product of Functions

Statement : If $y = u \cdot v$, where u and v are differentiable functions of x, then,

$$\frac{dy}{dx} = \frac{d}{dx}[u \cdot v]$$

$$= u \cdot \frac{dv}{dx} + v \frac{du}{dx}$$

This is known as product rule of derivatives.

Deduction - I :

If u is a differentiable function of x and k is a scalar multiple, then

$$\frac{d}{dx}(ku) = k \cdot \frac{du}{dx} + u \cdot \frac{dk}{dx}$$

$$= k \cdot \frac{du}{dx} \quad \ldots \because \frac{d}{dx}(k) = 0 \quad \therefore \text{ Second part = zero}$$

Deduction - II :

If u, v, w are differentiable functions of x and $y = u \cdot v \cdot w$, then

$$\frac{dy}{dx} = k \cdot \frac{d}{dx}[uvw]$$

$$= vw \cdot \frac{du}{dx} + uw \frac{dv}{dx} + uv \frac{dw}{dx}$$

Thus, we take derivative of a function keeping other two functions constant till the derivative of all functions in turn.

Derivative of Quotient of Functions

Statement :

If $y = \frac{u}{v}$, $v \neq 0$, where u and v are differentiable functions of x, then,

$$\frac{dy}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

$$= \frac{(\text{Denominator}) \times \left(\begin{matrix}\text{Derivative} \\ \text{of numerator}\end{matrix}\right) - (\text{Numerator}) \times \left(\begin{matrix}\text{Derivative} \\ \text{of denominator}\end{matrix}\right)}{(\text{Denominator})^2}$$

Deduction :

If $y = \frac{1}{v}$, where u = 1 and v is a differentiable function of x then using above rule, we have,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{v}\right] = \frac{v \cdot \frac{d}{dx}(1) - (1)\frac{d}{dx}(v)}{v^2}$$

Here, $\frac{d}{dx}(1) = 0$

$$\therefore \frac{dy}{dx} = -\frac{1}{v^2} \cdot \frac{dv}{dx}$$

Thus, $\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2} \cdot \frac{dv}{dx}$ … where v is a function of x

It follows that

(i) if $y = \log_e p$, where p is a differentiable function of x, then,

$$\frac{dy}{dx} = \frac{1}{p} \cdot \frac{dp}{dx}$$

(ii) if $p = \log_e r$, where r is a differentiable function of x, then,

$$\frac{dp}{dx} = \frac{1}{r} \cdot \frac{dr}{dx}$$

(iii) if $y = e^{\theta}$, where θ is a differentiable function of t, then,

$$\frac{dy}{dt} = e^{\theta} \cdot \frac{d\theta}{dt}$$ … and so on.

Derivative of Composite Functions (Chain Rule of Derivatives) :

Let $y = f(u)$ and $u = g(x)$.

Therefore, $y = f[g(x)]$, which is also written as $(fog)(x)$. This is a **function of a function** or a **composite function**.

We will state and prove the theorem for finding the derivative of such composite functions. This theorem is known as **chain rule** of derivatives.

Statement :

If y is a differentiable function of u and u is a differentiable function of x such that y is a composite function of x, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Deduction :

This chain rule of derivatives can be extended for double composite functions as below :

If y is a function of u, u is a function of v and v is a function of x, then we have :

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right) \times \left(\frac{du}{dv}\right) \times \left(\frac{dv}{dx}\right)$$

Notes :

(1) The single composite function can also be stated as below :

If $\frac{d}{dx}[f(x)] = f'(x)$ and $\frac{d}{dx}[g(x)] = g'(x)$, then

$$\frac{d}{dx}[f\{g(x)\}] = f'\{g(x)\} \times \frac{d}{dx}[g(x)] = f'\{g(x)\} \times g'(x)$$

(2) Using above concept of derivative of composite functions, we shall write some frequently used composite derivatives as below :

List of Composite Derivatives :

	Standard functions	Composite linear functions	Composite general functions
1.	$\frac{d}{dx}(x) = 1$	$\frac{d}{dx}(ax + b) = a$	$\frac{d}{dx}[f(x)] = f'(x)$
2.	$\frac{d}{dx}[x^n] = n \cdot x^{n-1}$	$\frac{d}{dx}[(ax + b)^n] = n(ax + b)^{n-1} \cdot a$	$\frac{d}{dx}[f^n(x)] = n f^{x-1}(x) \cdot f'(x)$
3.	$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{-1}{x^2}$	$\frac{d}{dx}\left[\frac{1}{ax + b}\right] = \frac{-1}{(ax + b)^2} \cdot a$	$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{-1}{f^2(x)} \cdot f'(x)$
4.	$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$	$\frac{d}{dx}\left[\sqrt{ax + b}\right] = \frac{1}{2\sqrt{ax + b}} \cdot a$	$\frac{d}{dx}\left[\sqrt{f(x)}\right] = \frac{1}{2\sqrt{f(x)}} \cdot f'(x)$
5.	$\frac{d}{dx}[\log_e x] = \frac{1}{x}$	$\frac{d}{dx}[\log_e (ax + b)] = \frac{1}{ax + b} \cdot a$	$\frac{d}{dx}[\log_e f(x)] = \frac{1}{f(x)} \cdot f'(x)$
6.	$\frac{d}{dx}[e^x] = e^x$	$\frac{d}{dx}[e^{ax + b}] = e^{ax + b} \cdot a$	$\frac{d}{dx}[e^{f(x)}] = e^{f(x)} \cdot f'(x)$
7.	$\frac{d}{dx}[a^x] = a^x \cdot \log_e a$	$\frac{d}{dx}[A^{ax + b}] = A^{ax + b} \cdot \log_e A \cdot a$	$\frac{d}{dx}[A^{f(x)}] = A^{f(x)} \cdot \log_e A \cdot f'(x)$
8.	$\frac{d}{dx}[\sin x] = \cos x$	$\frac{d}{dx}[\sin (ax + b)] = \cos (ax + b) \cdot a$	$\frac{d}{dx}[\sin f(x)] = \cos f(x) \cdot f'(x)$
9.	$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1 + x^2}$	$\frac{d}{dx}[\tan^{-1} (ax + b)] = \frac{1}{1 + (ax + b)^2} \cdot a$	$\frac{d}{dx}[\tan^{-1} f(x)] = \frac{1}{1 + f^2(x)} \cdot f'(x)$

Integration :

Integration is the reverse process of differentiation. It is used to find the accumulated quantity or the area under a curve. The result of an integration is called an integral, and the process itself is referred to as finding the integral of a function. The integral of a function f (x) with respect to x is written as

$$I = \int f(x) \cdot dx$$

where, dx represents that we are integrating with respect to the variable x. The integral sign '$\int$' represents the process of finding the area under the curve of the function.

There are two main types of integrals :

(1) Indefinite Integral : This represents the antiderivative of a function, often denoted as $\int f(x)\, dx + c$, where 'c' is the constant of integration. The result is a family of functions and it provides a way to find functions whose derivatives are known.

(2) Definite Integral : This calculates the area under a curve between two specific points. It is written as $\int_a^b f(x)\,dx$. It gives a numerical value that represents the net area between the curve and the x-axis from x = a to x = b.

Integration has numerous applications, including calculating areas, volumes, work and solving differential equations.

Together, derivatives and integrals are powerful tools in calculus, enabling us to understand and analyze how quantities change and accumulative in a wide range of real-world scenarios in science, engineering, economics and more.

1.2 PARTIAL DERIVATIVE (TWO VARIABLES)

1.2.1 Introduction

Partial derivatives are a fundamental concept in multivariable calculus, which is an extension of calculus that deals with functions of multiple variables. Unlike ordinary derivatives, which focus on how a function changes with respect to a single variable, partial derivatives allow us to understand how a function changes concerning one specific variable while holding all other variables constant. They are crucial in various fields of science, engineering, economics and more where quantities depend on multiple factors.

In mathematics, the process of examining the slope of a surface in only one direction at a time is called partial differentiation. In other words, a partial derivative allows only one variable to vary (change) at a time and helps us to analyze surfaces for minimum or maximum points.

Consider a function u = f (x, y) where u is the dependent variable and x and y are independent variables.

The partial derivative of u = f (x, y) w.r.to x is the ordinary derivative of u w.r.to x, keeping y constant. It is denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or u_x or f_x and is known as first-order partial derivate of u w.r.t. x.

Mathematically,

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

Similarly, the partial derivative of u = f (x, y) w.r.to y is the ordinary derivative of u w.r.to y treating x as constant. It is denoted by $\frac{\partial u}{\partial y}$, or $\frac{\partial f}{\partial y}$ or u_y or f_y and is known as first-order partial derivative of u w.r.t. y.

Mathematically,

$$\frac{\partial u}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

For functions of more than two variables, you can compute partial derivatives with respect to each individual variable.

Partial derivatives provide crucial information about how a function behaves in a multidimensional space. Partial derivatives are extensively used in various fields like Physics, Engineering, Economics, Computer Science, Environmental Science etc. Thus, partial derivatives are a foundational concept in calculus and a vital tool for understanding and solving problems that involve functions of multiple variables.

1.2.2 Partial Derivative of First-Order, Second-Order and Mixed-Order

Partial derivatives are a fundamental concept in multivariable calculus, allowing us to determine how a function changes with respect to one specific variable while keeping others constant. There are three main types of partial derivatives as discussed below :

(1) First-Order Partial Derivatives :

The first-order partial derivative of a function f (x, y) with respect to one of its variable, say x, is denoted as $\frac{\partial f}{\partial x}$ or $\frac{\partial}{\partial x} f$. It measures the rate of change of f with respect to x while holding y constant.

Mathematically, the first-order partial derivative is defined as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

Similarly, you can compute the first-order partial derivative with respect to y, denoted as $\frac{\partial f}{\partial y}$.

(2) Second-Order Partial Derivative :

The second-order partial derivative measures how the first-order partial derivative changes with respect to another variable. There are three possible second-order partial derivatives for a function f (x, y).

(i) $\frac{\partial^2 f}{\partial x^2}$ which measures how the rate of change of f with respect to x changes concerning x.

(ii) $\frac{\partial^2 f}{\partial y^2}$ which measures how the rate of change of f with respect to y changes concerning y.

(iii) $\frac{\partial^2 f}{\partial x\, \partial y}$ or $\frac{\partial^2 f}{\partial y\, \partial x}$ which measures how the rate of change of f with respect to x changes concerning y or vice versa.

Mathematically, these second-order partial derivatives are calculated by taking the partial derivative of the first-order partial derivatives.

For example,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

These are called second-order partial derivatives. Similarly, other higher-order derivatives can also be obtained.

Remarks :

(i) If f (x, y) possesses continuous second-order partial derivatives $\frac{\partial^2 f}{\partial x\, \partial y}$ and $\frac{\partial^2 f}{\partial y\, \partial x}$ then $\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}$. This is called commutative property.

(ii) Standard rules of differentiation of sum, difference, product and quotient are also applicable for partial differentiation.

(3) Mixed-Order Partial Derivative :

Mixed-order partial derivatives involve taking a partial derivative with respect to one variable and then another. For a function f (x, y), you can have mixed-order partial derivatives like $\frac{\partial^2 f}{\partial x\, \partial y}$ or $\frac{\partial^2 f}{\partial y\, \partial x}$. These indicate how the rate of change of f with respect to one variable changes concerning another variable.

Mathematically, the mixed-order partial derivatives are obtained by taking the partial derivative of the first-order partial derivatives with respect to the other variable. For example,

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

These derivatives are crucial in various fields of science and engineering, including physics, economics and optimization, as they provide insights into how functions behave in multivariable contexts. They are used extensively in solving partial differential equations which model many physical and natural phenomena.

Solved Examples

Example 1.1 : If $f(x, y) = x^3y^2$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Solution : Given : $f(x, y) = x^3y^2$

Differentiating partially w.r.to x, keeping y constant, we have,

$\therefore$

$$\frac{\partial f}{\partial x} = y^2 \cdot \frac{\partial}{\partial x}(x^3)$$

$$= y^2 \cdot 3x^2$$

$$= 3x^2y^2$$

Similarly differentiating partially w.r.to y, keeping x constant, we have,

$$\therefore \quad \frac{\partial f}{\partial y} = x^3 \cdot \frac{\partial}{\partial y}(y^2)$$
$$= x^3 \cdot 2y$$
$$= 2x^3y$$

Example 1.2 : If $f(x, y) = 2x + 3y$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Solution : Given : $f(x, y) = 2x + 3y$

Differentiating partially w.r.to x, keeping y constant, we have,

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2x) + 3y\frac{\partial}{\partial x}(1)$$
$$= 2 + 0$$
$$= 2$$

Similarly differentiating partially w.r.to y, keeping x constant, we have,

$$\therefore \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2x) + 3\frac{\partial}{\partial y}(y)$$
$$= 0 + 3(1)$$
$$= 3$$

Example 1.3 : If $f(x, y) = (x^2 + 1)(2y - 1)$ find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Solution : Given : $f(x, y) = (x^2 + 1)(2y - 1)$

Differentiating partially w.r.to x, using product rule, we have,

$$\therefore \quad \frac{\partial f}{\partial x} = (x^2 + 1) \cdot \frac{\partial}{\partial x}(2y - 1) + (2y - 1) \cdot \frac{\partial}{\partial x}(x^2 + 1)$$
$$= (x^2 + 1) \cdot (0) + (2y - 1)(2x + 0)$$
$$= 0 + (2y - 1) \cdot 2x$$
$$= 2x(2y - 1)$$

Similarly differentiating partially w.r.to y, we get,

$$\therefore \quad \frac{\partial f}{\partial y} = (x^2 + 1) \cdot \frac{\partial}{\partial y}(2y - 1) + (2y - 1) \cdot \frac{\partial}{\partial y}(x^2 + 1)$$
$$= (x^2 + 1) \cdot (2) + (2y - 1)(0)$$
$$= 2(x^2 + 1)$$

Example 1.4 : If $f(x, y) = \frac{xy}{x^2 + y^2}$ find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Solution : Given : $f(x, y) = \frac{xy}{x^2 + y^2}$

Differentiating partially w.r.to x, using quotient rule, we have,

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{(x^2 + y^2) \cdot \frac{\partial}{\partial x}(xy) - xy \cdot \frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2} \quad \ldots \text{(y is constant)}$$
$$= \frac{(x^2 + y^2) \cdot y - xy(2x)}{(x^2 + y^2)^2}$$
$$= \frac{x^2y + y^3 - 2x^2y}{(x^2 + y^2)^2}$$
$$= \frac{y^3 - x^2y}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$

Similarly differentiating partially w.r.to y, we get,

$$\therefore \quad \frac{\partial f}{\partial y} = \frac{(x^2 + y^2) \cdot \frac{\partial}{\partial y}(xy) - xy \cdot \frac{\partial}{\partial y}(x^2 + y^2)}{(x^2 + y^2)^2} \quad \ldots \text{(x is constant)}$$

$$= \frac{(x^2 + y^2) \cdot x - xy\,(2y)}{(x^2 + y^2)^2}$$

$$= \frac{x^3 + xy^2 - 2xy^2}{(x^2 + y^2)^2}$$

$$= \frac{x^3 - xy^2}{(x^2 + y^2)^2} = \frac{x\,(x^2 - y^2)}{(x^2 + y^2)^2}$$

Example 1.5 : If $f(x, y) = e^{x+y+2}$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Solution : Given :

$$f(x, y) = e^{x+y+2}$$

$$= e^x \cdot e^y \cdot e^2$$ Where, e^2 = constant

Note the use of Laws of Indices

Differentiating partially w.r.to x, keeping y constant, we have,

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^x \cdot e^y \cdot e^2)$$

$$= e^y \cdot e^2 \cdot \frac{\partial}{\partial x}(e^x)$$

$$= e^y \cdot e^2 \cdot e^x \cdot (1)$$

$$= e^x \cdot e^y \cdot e^2 = e^{x+y+2}$$

Similarly differentiating partially w.r.to y, keeping x constant, we have,

$$\therefore \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(e^x \cdot e^y \cdot e^2)$$

$$= e^x \cdot e^2 \cdot \frac{\partial}{\partial y}(e^y)$$

$$= e^x \cdot e^2 \cdot e^y \cdot (1)$$

$$= e^{x+y+2}$$

Example 1.6 : If $f(x, y) = e^{\log(xy)}$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Solution : Given : $f(x, y) = e^{\log(xy)}$

Differentiating partially w.r.t. x, keeping y constant, we have,

$$\therefore \quad \frac{\partial f}{\partial x} = e^{\log(xy)} \cdot \frac{\partial}{\partial x}[\log(xy)]$$

$$= e^{\log(xy)} \cdot \frac{1}{xy} \cdot \frac{\partial}{\partial x}(xy)$$

$$= e^{\log(xy)} \cdot \frac{1}{xy} \cdot y \cdot \frac{\partial}{\partial x}(x)$$ … (y is constant)

$$= e^{\log(xy)} \cdot \frac{1}{xy} \cdot y \cdot (1)$$

$$= \frac{1}{x} \cdot e^{\log(xy)}$$

Similarly differentiating partially w.r.to y, keeping x constant, we have,

$$\therefore \quad \frac{\partial f}{\partial y} = e^{\log(xy)} \cdot \frac{\partial}{\partial y}[\log(xy)]$$

$$= e^{\log(xy)} \cdot \frac{1}{xy} \cdot \frac{\partial}{\partial y}(xy)$$

$$= e^{\log(xy)} \cdot \frac{1}{xy} \cdot x \cdot \frac{\partial}{\partial y}(y)$$ … (x is constant)

$$= e^{\log(xy)} \cdot \frac{1}{xy} \cdot x \cdot (1)$$

$$= \frac{1}{y} \cdot e^{\log(xy)}$$

Example 1.7 : If $f(x, y) = x^2y^2 + \sin x + \cos y$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Solution : Given : $f(x, y) = x^2y^2 + \sin x + \cos y$

Differentiating partially w.r.to x, keeping y constant, we have,

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^2) + \frac{\partial}{\partial x}(\sin x) + \frac{\partial}{\partial x}(\cos y)$$

$$= y^2 \cdot \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(\sin x) + \cos y \cdot \frac{\partial}{\partial x}(1)$$

$$= y^2 \cdot (2x) + \cos x + \cos y\,(0)$$

$$= 2xy^2 + \cos x$$

Similarly differentiating partially w.r.to y, keeping x constant, we get,

$$\therefore \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y^2) + \frac{\partial}{\partial y}(\sin x) + \frac{\partial}{\partial y}(\cos y)$$

$$= x^2 \cdot \frac{\partial}{\partial y}(y^2) + \sin x \cdot \frac{\partial}{\partial y}(1) + (-\sin y)$$

$$= x^2 \cdot 2y + \sin x \cdot (0) - \sin y$$

$$= 2x^2y + \sin y$$

Example 1.8 : If $f(x, y) = x^y$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Solution : Given : $f(x, y) = x^y$

Differentiating partially w.r.to x, keeping y constant, we have,

$$\therefore \quad \frac{\partial f}{\partial x} = y \cdot x^{y-1}$$

Similarly differentiating partially w.r.to y, keeping x constant, we have,

$$\therefore \quad \frac{\partial f}{\partial y} = x^y \cdot \log_e x$$

Example 1.9 : If $f(x, y) = \frac{1}{xy}$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Solution : Given : $f(x, y) = \frac{1}{xy}$

Differentiating partially w.r.to x, keeping y constant, we get,

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{-1}{(xy)^2} \cdot \frac{\partial}{\partial x}(xy)$$

$$= \frac{-1}{x^2y^2} \cdot y \cdot (1) = \frac{-1}{x^2y}$$

Similarly differentiating partially w.r.to y, keeping x constant, we have,

$$\therefore \quad \frac{\partial f}{\partial y} = \frac{-1}{(xy)^2} \cdot \frac{\partial}{\partial y}(xy) \qquad \ldots \text{(x is constant)}$$

$$= \frac{-1}{x^2y^2} \cdot x \cdot (1) = \frac{-1}{xy^2}$$

Example 1.10 : If $f(x, y) = \log_e(x^2 + y^2)$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Solution : Given : $f(x, y) = \log_e(x^2 + y^2)$

Differentiating partially w.r.to x, keeping y constant, we have,

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial x}(x^2 + y^2)$$

$$= \frac{1}{x^2 + y^2} \cdot (2x + 0)$$

$$= \frac{2x}{x^2 + y^2}$$

Similarly differentiating partially w.r.to y, keeping x constant, we have,

$$\therefore \quad \frac{\partial f}{\partial y} = \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial y}(x^2 + y^2)$$
$$= \frac{1}{x^2 + y^2} \cdot (0 + 2y)$$
$$= \frac{2y}{x^2 + y^2}$$

Example 1.11 : If $f(x, y) = \sin(xy) + xy$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Solution : Given : $f(x, y) = \sin(xy) + xy$

Differentiating partially w.r.to x, keeping y constant, we have,

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\sin(xy)] + \frac{\partial}{\partial x}(xy)$$
$$= \cos(xy) \cdot \frac{\partial}{\partial x}(xy) + y \cdot \frac{\partial}{\partial x}(x)$$
$$= \cos(xy) \cdot y \cdot (1) + y(1)$$
$$= y \cdot \cos(xy) + y$$

Similarly differentiating partially w.r.to y, keeping x constant, we get,

$$\therefore \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sin(xy)] + \frac{\partial}{\partial y}(xy)$$
$$= \cos(xy) \cdot \frac{\partial}{\partial y}(xy) + x \cdot \frac{\partial}{\partial y}(y)$$
$$= \cos(xy) \cdot x \cdot (1) + x(1)$$
$$= x \cdot \cos(xy) + x$$

Example 1.12 : If $f(x, y) = \sqrt{1 + x^2 + y^2}$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Solution : Given : $f(x, y) = \sqrt{1 + x^2 + y^2}$

Differentiating partially w.r.to x, keeping y constant, we have,

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{1}{2\sqrt{1 + x^2 + y^2}} \cdot \frac{\partial}{\partial x}(1 + x^2 + y^2)$$
$$= \frac{1}{2\sqrt{1 + x^2 + y^2}} \cdot (0 + 2x + 0)$$
$$= \frac{2x}{2\sqrt{1 + x^2 + y^2}}$$
$$= \frac{x}{\sqrt{1 + x^2 + y^2}}$$

Similarly differentiating partially w.r.to y, keeping x constant, we have,

$$\therefore \quad \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{1 + x^2 + y^2}} \cdot \frac{\partial}{\partial y}(1 + x^2 + y^2)$$
$$= \frac{1}{2\sqrt{1 + x^2 + y^2}} \cdot (0 + 0 + 2y)$$
$$= \frac{2y}{2\sqrt{1 + x^2 + y^2}}$$
$$= \frac{y}{\sqrt{1 + x^2 + y^2}}$$

Example 1.13 : If $f(x, y) = x^2y^3$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point P (1, 2).

Solution : Given : $f(x, y) = x^2y^3$

Differentiating partially w.r.to x, keeping y constant, we have,

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^3)$$

$$= y^3 \cdot \frac{\partial}{\partial x}(x^2) = y^3 \cdot (2x) = 2xy^3$$

$$\therefore \quad \frac{\partial f}{\partial x} \text{ at } (1, 2) = 2\,(1)\,(2)^3$$

$$= 16$$

Similarly differentiating partially w.r.to y, keeping x constant, we have,

$$\therefore \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y^3)$$

$$= x^2 \cdot \frac{\partial}{\partial y}(y^3)$$

$$= x^2 \cdot 3y^2$$

$$= 3x^2y^2$$

$$\therefore \quad \frac{\partial f}{\partial y} \text{ at } (1, 2) = 3\,(1)^2 \cdot (2)^2$$

$$= 12$$

Example 1.14 : If f(x, y) = x sin y + y sin x + xy. Find $\frac{\partial f}{\partial x}$. **(S-23) [2 Marks]**

Solution : Given : f(x, y) = x sin y + y sin x + xy

Differentiating partially w.r.t. x, keeping y constant.

$$\frac{\partial f}{\partial x} = \sin y \frac{\partial}{\partial x} x + y \frac{\partial}{\partial x} \sin x + y \frac{\partial}{\partial x} x$$

$$= \sin y + y \cos x + y$$

Example 1.15 : If f(x, y) = x^4 sin y + y^4 cos x + x^3. Find $\frac{\partial^2 f}{\partial x \partial y}$. **(S-23) [4 Marks]**

Solution : Given : f(x, y) = x^4 sin y + y^4 cos x + x^3

Differentiating partially w.r.t. x, keeping y

$$\frac{\partial f}{\partial x} = \sin y \frac{\partial}{\partial x} x^4 + y^4 \frac{\partial}{\partial x} \cos x + \frac{\partial}{\partial x} x^3$$

$$\frac{\partial f}{\partial x} = \sin y\,(4x^3) + y^4\,(-\sin x) + 3x^2$$

$$\frac{\partial f}{\partial x} = 4x^3 \sin y - \sin x y^4 + 3x^2 \quad \ldots (1)$$

Differentiating equation (1) w.r.t. y keeping x constant.

$$\frac{\partial^2 f}{\partial x \partial y} = 4x^3 \frac{\partial}{\partial y} \sin y - \sin x \frac{\partial}{\partial y} y^4 + \frac{\partial}{\partial y} 3x^2$$

$$= 4x^3 \cos y - \sin x\,(4y^3) + 0$$

$$= 4x^3 \cos y - 4y^3 \sin x$$

Example 1.16 : If $z = e^{-et} \sin \theta$. Find $\frac{\partial^2 z}{\partial t\, \partial \theta}$. **(S-24)**

Solution: Let $z = e^{-t} \sin \theta$

Partial differentiation w.r.t. θ, $\frac{\partial z}{\partial \theta} = e^{-t} \cos \theta$

$$\therefore \quad \frac{\partial^2 z}{\partial t\, \partial \theta} = -e^{-t} \cos \theta$$

Example 1.17 : If $z = 5x^4 + 2x^3y^2 - 3y$. Find $\frac{\partial z}{\partial x}$. **(S-24)**

Solution: $z = 5x^4 + 2x^3y^2 - 3y$

Partial derivative w.r.t. x, $\frac{\partial z}{\partial x} = 5(4x^3) + 2(3x^2)\,y^2 - 0$

$$\frac{\partial z}{\partial x} = 20\,x^3 + 6x^2y^2$$
