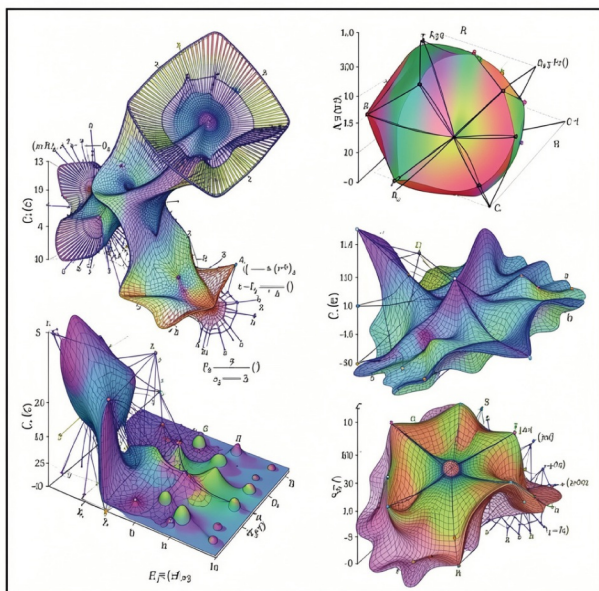


Analysis

Foundations of Differential and Integral Calculus



Lucien Sina

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Chapter 1

Preface

Analysis is one of the fundamental areas of mathematics and is also of central importance for many other mathematically oriented fields of study. In particular, for students of mathematics, computer science, physics, and engineering, it provides an indispensable foundation for further study.

This book is aimed primarily at students in their first two semesters, especially at those who encounter analysis systematically for the first time in their first semester. Its focus is a careful introduction to the central concepts and methods of analysis. These include, among others, mathematical induction, the properties of the real numbers, sequences and series, continuity, differentiation, integration, as well as more advanced topics such as uniform convergence, power series, and Taylor series.

The presentation places particular emphasis on math-

ematical precision. Concepts are introduced exactly, connections are developed carefully, and statements are proved consistently. At the same time, care has been taken to present the material not only formally, but also in an intuitive and comprehensible way. Many examples, remarks, and exercises are intended to help readers not merely read the material, but truly understand it.

The book is structured so that it can be used both alongside a lecture course and for independent study. Detailed derivations and numerous exercises with solutions are intended to make analysis more accessible and to enable readers to work through and consolidate the material systematically.

My aim was to write a textbook that takes the mathematical content seriously without making it seem unnecessarily intimidating. For many students, analysis is at first a demanding subject, but with growing understanding it gains clarity and structure. If this book helps to make that access easier and perhaps even conveys some enjoyment of mathematics, then it has fulfilled its purpose.

Lucien Sina

Chapter 2

Mathematical Induction and Combinatorics

Analysis deals with continuous change, limiting processes, series, sequences, functions, and the question of how a general law for infinitely many cases can arise from finitely many steps. For precisely this reason, it is helpful to begin the study of analysis with a proof principle that formalizes exactly this kind of step-by-step reasoning: mathematical induction.

With its help, statements about all natural numbers can be proved by first establishing the initial case and then showing that the validity of any given case implies the validity of the next one. In this way, a single secure starting point leads to a universally valid conclusion. This

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principle appears repeatedly in analysis, for instance in summation formulas, inequalities, recursions, estimates for sequences, and many other statements that involve a natural number as a parameter.

Mathematical induction is therefore not merely a technical tool, but a central way of thinking in mathematics. Anyone who masters it reliably has an important foundation for understanding and developing analysis.

2.1 Proof by Mathematical Induction

The principle of mathematical induction is based on a fundamental property of the natural numbers: Every natural number is either the beginning of the sequence or is obtained from a smaller natural number by adding 1. This structure is exactly what makes it possible to derive statements for all natural numbers from an initial case and an inheritance step.

Definition 1 (Principle of mathematical induction). *Let $M \subseteq \mathbb{N}_0$. If*

1. $0 \in M$, and
2. *for all $n \in \mathbb{N}_0$, the implication holds: if $n \in M$, then $n + 1 \in M$,*

then already $M = \mathbb{N}_0$.

This principle can also be formulated as a general statement about properties:

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Definition 2 (Induction principle in statement form).
Let $P(n)$ be a statement about $n \in \mathbb{N}_0$. If

1. *$P(0)$ is true, and*
2. *for every $n \in \mathbb{N}_0$, the validity of $P(n)$ implies the validity of $P(n + 1)$,*

then $P(n)$ is true for all $n \in \mathbb{N}_0$.

A proof by mathematical induction therefore always consists of two essential parts:

1. **Base case:** One shows that the statement holds for the first relevant value of the natural number.
2. **Induction step:** One assumes that the statement holds for an arbitrary but fixed n , and then shows that it also holds for $n + 1$.

The induction step is the decisive transition. It shows not only that a single case is true, but that the truth of the statement propagates along the natural numbers. Once the starting point has been secured, every subsequent case is automatically covered.

This is called *complete* induction because individual numbers are not checked in isolation; instead, the entire infinite system of natural numbers is captured by a general logical structure. In analysis, this method is particularly important, since many statements must be proved for all $n \in \mathbb{N}$ or $n \in \mathbb{N}_0$.

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Remark 3. *Depending on the convention, the natural numbers begin with 0 or with 1. The form used here, $\mathbb{N}_0$, contains 0. If a later chapter works with $\mathbb{N} = \{1, 2, 3, \dots\}$, only the base case changes, not the underlying principle.*

The principle of complete induction reveals its strength especially when it is applied to a concrete statement. A classical example is the sum of the first n natural numbers. This example makes it particularly clear how a proof by induction is structured and why the transition from n to $n + 1$ forms the real core of the method.

Example 4 (Small Gaussian summation formula). *One of the first and at the same time best-known applications of mathematical induction is the summation formula*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad (n \in \mathbb{N}).$$

It describes the sum of the first n natural numbers and is therefore often called the small Gaussian summation formula.

At first glance, the left-hand side seems to require a new calculation for each new value of n . The formula shows, however, that this sum can be transformed into a simple closed form. Statements of this kind are typical in analysis: from a sequence of individual values, a general law emerges.

Proof. We prove the formula by mathematical induction on n .

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For clarity, we denote the sum of the first n natural numbers by

$$S_n := 1 + 2 + \cdots + n.$$

Thus we have to show that

$$S_n = \frac{n(n+1)}{2} \quad \text{for all } n \in \mathbb{N}.$$

1. Base case. For $n = 1$, we have

$$S_1 = 1.$$

On the other hand, the right-hand side gives

$$\frac{1 \cdot 2}{2} = 1.$$

Hence the statement is true for the initial value $n = 1$.

2. Induction step. Now assume that the formula already holds for a fixed but arbitrary $n \in \mathbb{N}$. That is,

$$S_n = \frac{n(n+1)}{2}.$$

This assumption is called the *induction hypothesis*. It does not mean that we want to prove the statement only for this single value of n ; rather, we assume it as already known in order to derive the next case from it.

We now consider the sum up to $n + 1$:

$$S_{n+1} = 1 + 2 + \cdots + n + (n + 1).$$

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Since $1 + 2 + \cdots + n = S_n$, the induction hypothesis gives

$$S_{n+1} = S_n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1).$$

Now we factor out $n + 1$:

$$S_{n+1} = (n + 1) \left(\frac{n}{2} + 1 \right) = (n + 1) \left(\frac{n + 2}{2} \right).$$

Thus

$$S_{n+1} = \frac{(n + 1)(n + 2)}{2}.$$

This is exactly the desired formula for the case $n + 1$.

Since the base case has been proved and validity for n always implies validity for $n + 1$, the statement is proved for all $n \in \mathbb{N}$. $\square$

This example illustrates the basic pattern of complete induction particularly clearly: first, a starting point is established, and then it is shown that each case implies the next one. In this way, a statement can be proved not merely for some numbers, but for all natural numbers.

2.2 Summation Notation

When many terms are added in analysis, it would be impractical to write out every sum in full. For this reason, a special notation for sums is used, the so-called *summation notation*. It makes it possible to write long additions in a clear and general form.

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Definition 5 (Summation notation). *Let $a_1, a_2, \dots, a_n$ be numbers and let $n \in \mathbb{N}$. Then*

$$\sum_{i=1}^n a_i := a_1 + a_2 + \cdots + a_n.$$

The number i is called the running index or summation index. It serves only to organize the individual summands and is not regarded as an independent quantity within the sum.

With the help of summation notation, many statements can be formulated very compactly. Sums whose terms depend on a natural number occur especially often. Precisely in such situations, complete induction is a particularly suitable method of proof.

In the following, we prove a classical example that also plays an important role in working with sums.

Example 6 (Sum of the first odd numbers). *For all $n \in \mathbb{N}$, one has*

$$\sum_{i=1}^n (2i - 1) = n^2.$$

In other words, the sum of the first n odd numbers is always equal to the square of n .

Proof. We prove the statement by complete induction on n .

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Base case: For $n = 1$, we have

$$\sum_{i=1}^1 (2i - 1) = 2 \cdot 1 - 1 = 1$$

and at the same time

$$1^2 = 1.$$

Thus the assertion is true for $n = 1$.

Induction hypothesis: Suppose that, for a fixed but arbitrary $n \in \mathbb{N}$, the statement already holds, that is,

$$\sum_{i=1}^n (2i - 1) = n^2.$$

Induction step: We now show that the statement then also holds for $n + 1$. To this end, consider

$$\sum_{i=1}^{n+1} (2i - 1).$$

We separate off the last summand:

$$\sum_{i=1}^{n+1} (2i - 1) = \sum_{i=1}^n (2i - 1) + (2(n + 1) - 1).$$

Using the induction hypothesis, it follows that

$$\sum_{i=1}^{n+1} (2i - 1) = n^2 + 2n + 1.$$

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Since

$$n^2 + 2n + 1 = (n + 1)^2$$

we obtain

$$\sum_{i=1}^{n+1} (2i - 1) = (n + 1)^2.$$

This proves the induction step.

Consequently,

$$\sum_{i=1}^n (2i - 1) = n^2$$

for all $n \in \mathbb{N}$. □

Complete induction and many of the examples considered so far are based on the natural numbers. We therefore now introduce the most important number sets, which will be used constantly throughout analysis.

Definition 7 (Natural numbers). *The set of natural numbers is*

$$\mathbb{N} := \{1, 2, 3, \dots\}.$$

If one also includes the number 0, one obtains the set

$$\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}.$$

The natural numbers are used in particular for counting and ordering. Because of their step-by-step structure, they form the basis of complete induction.

In order to form differences of natural numbers without restriction, the natural numbers are extended to the integers.

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Definition 8 (Integers). *The set of integers is*

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

Thus it contains the positive and the negative whole numbers as well as 0.

Remark 9. *Complete induction is a proof method and not a definition of the natural numbers. However, it is closely connected with the natural numbers, since they are arranged in a successive sequence-like structure. It is precisely this structure that makes the transition from n to $n + 1$ possible in the induction step.*

Further number sets will be introduced later, in particular the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. For the foundations of complete induction, however, the natural numbers are sufficient at first.

2.3 Product Notation and Factorials

Product notation and factorials are not only useful abbreviations for repeated multiplication, but also central concepts in combinatorics. This area of mathematics studies how many possibilities certain arrangements and selection processes allow. The factorial arises naturally in this context, because it describes the number of all possible orderings of a finite set. Just as summation notation represents long additions compactly, the *product notation* is used to write longer multiplications clearly

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and concisely. It plays an important role both in analysis and in combinatorics.

Definition 10 (Product notation). *Let $a_1, a_2, \dots, a_n$ be numbers and let $n \in \mathbb{N}$. Then*

$$\prod_{i=1}^n a_i := a_1 \cdot a_2 \cdot \dots \cdot a_n.$$

The number i is again called the running index or multiplication index. It serves only to arrange the factors in order.

Analogously to sums, one also considers the empty product. It is useful to define it as 1, since 1 is the neutral element of multiplication.

Remark 11 (Empty product). *For $n = 0$, one sets*

$$\prod_{i=1}^0 a_i := 1.$$

This convention is not arbitrary; rather, it makes many formulas valid in a uniform way and without exceptional cases.

A particularly important application of product notation is the *factorial*.

Definition 12 (Factorial). *For $n \in \mathbb{N}_0$, the factorial is defined by*

$$n! := \prod_{k=1}^n k.$$

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In particular, this gives

$$1! = 1, \quad 2! = 1 \cdot 2, \quad 3! = 1 \cdot 2 \cdot 3, \quad \dots$$

and, because of the convention for the empty product, also

$$0! = 1.$$

In combinatorics, the factorial counts the number of all possible arrangements of n distinct elements.

Theorem 13 (Arrangements of an n -element set). *Let M be a set with exactly n elements. Then there are exactly $n!$ different arrangements of the elements of M .*

Proof. We justify the statement step by step.

For the first element of an arrangement, there are n possibilities, since each of the n elements may stand in first position. Once the first element has been chosen, $n - 1$ elements remain, so there are $n - 1$ possibilities for the second position. For the third position, $n - 2$ possibilities remain, and so on.

Continuing in this way up to the last position, one obtains altogether

$$n \cdot (n - 1) \cdot (n - 2) \cdots 2 \cdot 1 = n!.$$

Thus an n -element set has exactly $n!$ different arrangements. $\square$

Example 14. *A set with three elements has exactly*

$$3! = 3 \cdot 2 \cdot 1 = 6$$

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different arrangements. If the elements are, for instance, a, b, c , then these arrangements are

$$abc, acb, bac, bca, cab, cba.$$

Remark 15. *The factorial grows very quickly. Already for $n = 5$, one has $5! = 120$, and for larger numbers the values soon become very large. This is precisely why the factorial is important in analysis, for example in series, estimates, and later also in Taylor expansions.*

2.4 The Binomial Coefficient

In combinatorics, the same question arises again and again: How many ways are there to choose some elements from a finite set without taking their order into account? The binomial coefficient was introduced precisely for counting problems of this kind. It is one of the most important numerical concepts in elementary combinatorics and later plays a central role in analysis, for example in the binomial theorem and in series expansions.

Definition 16 (Binomial coefficient). *For $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ with $k \leq n$, the binomial coefficient is defined by*

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

For $k > n$, one additionally sets

$$\binom{n}{k} := 0.$$

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The binomial coefficient $\binom{n}{k}$ is read as “ n choose k .” It counts the number of ways to choose exactly k elements from an n -element set when the order of selection does not matter.

Theorem 17 (Pascal’s recursion formula). *For all $n \in \mathbb{N}$ and all $k \in \{1, \dots, n-1\}$, one has*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. We give a combinatorial proof.

Consider a set M with n elements and fix a particular element $x \in M$. We want to count all k -element subsets of M . By definition, this number is precisely $\binom{n}{k}$.

Now we divide all k -element subsets of M into two disjoint cases:

1. The subset contains the element x .
2. The subset does not contain the element x .

In the first case, in addition to x , one must choose $k-1$ further elements from the remaining $n-1$ elements. This is possible in $\binom{n-1}{k-1}$ ways.

In the second case, x is not chosen, so all k elements must come from the remaining $n-1$ elements. This is possible in $\binom{n-1}{k}$ ways.

Since the two cases are mutually exclusive and together cover all possibilities, the total number of k -

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element subsets is

$$\binom{n-1}{k-1} + \binom{n-1}{k}.$$

On the other hand, by definition this total number is $\binom{n}{k}$. Hence

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This proves the assertion. $\square$

Remark 18. *Pascal's formula is a recurrence relation: it expresses one binomial coefficient in terms of two neighboring binomial coefficients. It is therefore especially useful for constructing tables of binomial coefficients step by step.*

The combinatorial meaning of the binomial coefficient is shown by the following fundamental theorem. It explains why the expression

$$\binom{n}{k}$$

describes precisely the number of k -element subsets of an n -element set.

Theorem 19 (Number of k -element subsets). *Let M be a set with exactly n elements, and let $k \in \{0, 1, \dots, n\}$. Then there are exactly*

$$\binom{n}{k}$$

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distinct k -element subsets of M .

Proof. We count the k -element subsets of M .

First, we do not consider the subsets themselves, but ordered selections of k distinct elements from M . For the first element there are n possibilities, for the second still $n - 1$, then $n - 2$, and so on, until for the k -th element $n - k + 1$ possibilities remain. Altogether, there are therefore

$$n \cdot (n - 1) \cdot (n - 2) \cdots (n - k + 1) = \frac{n!}{(n - k)!}$$

such ordered selections.

However, each k -element subset is counted several times in this way, namely exactly as often as its k elements can be arranged. There are $k!$ different arrangements.

Therefore, one obtains the number of k -element subsets by dividing the number of ordered selections by $k!$:

$$\frac{1}{k!} \cdot \frac{n!}{(n - k)!} = \frac{n!}{k!(n - k)!}.$$

By the definition of the binomial coefficient, this is precisely

$$\binom{n}{k}.$$

Thus an n -element set has exactly $\binom{n}{k}$ distinct k -element subsets. □

2.5 The Binomial Theorem

When powers of a binomial $(a + b)$ are expanded, the resulting expressions become longer as the exponent increases. What is remarkable, however, is that these expressions are formed according to a very regular pattern. This pattern is described precisely by the binomial theorem. It is one of the most important formulas in algebra and also plays a central role in analysis, for example in series expansions and estimates.

If the corresponding coefficients are arranged row by row, one obtains *Pascal's triangle*. Each row contains the binomial coefficients, and each interior entry is obtained as the sum of the two entries above it. Pascal's recurrence formula explains exactly this rule.

Remark 20 (Pascal's triangle). *The binomial coefficients can be arranged in the following way:*

$$\begin{array}{cccc}
 & & \binom{0}{0} & \\
 & \binom{1}{0} & & \binom{1}{1} \\
 & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
 \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3}
 \end{array}$$

and so on.

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At the boundary one always has

$$\binom{n}{0} = \binom{n}{n} = 1,$$

and for the interior entries the recurrence formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad (1 \leq k \leq n-1).$$

These two properties completely describe Pascal's triangle.

After the binomial coefficients have been introduced as counting numbers, their second central meaning now appears: they are precisely the coefficients in the expansion of a power of $(a+b)$. The following theorem describes this connection.

Theorem 21 (Binomial theorem). *Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Then*

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$

Proof. We prove the statement by mathematical induction on n .

Base case: For $n = 1$, we have

$$(a+b)^1 = a+b.$$

On the other hand, the right-hand side gives

$$\sum_{k=0}^1 \binom{1}{k} a^{1-k} b^k = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 = a+b.$$

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Thus the statement is true for $n = 1$.

Induction hypothesis: Assume that, for a fixed but arbitrary $n \in \mathbb{N}$,

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$

Induction step: We now prove the claim for $n + 1$. To this end, we write

$$(a + b)^{n+1} = (a + b)^n (a + b).$$

Using the induction hypothesis, it follows that

$$(a + b)^{n+1} = \left(\sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \right) (a + b).$$

Expanding gives

$$(a + b)^{n+1} = \sum_{k=0}^n \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^n \binom{n}{k} a^{n-k} b^{k+1}.$$

In the second sum, we perform the change of index $k \mapsto k - 1$:

$$\sum_{k=0}^n \binom{n}{k} a^{n-k} b^{k+1} = \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n+1-k} b^k.$$

Hence we obtain

$$(a+b)^{n+1} = a^{n+1} + \sum_{k=1}^n \left(\binom{n}{k} + \binom{n}{k-1} \right) a^{n+1-k} b^k + b^{n+1}.$$

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By Pascal's recurrence formula, for $1 \leq k \leq n$,

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Moreover,

$$a^{n+1} = \binom{n+1}{0} a^{n+1} b^0 \quad \text{and} \quad b^{n+1} = \binom{n+1}{n+1} a^0 b^{n+1}.$$

Therefore,

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k.$$

This proves the induction step.

Thus the formula holds for all $n \in \mathbb{N}$. □

Remark 22. *The binomial theorem also explains why the binomial coefficients in Pascal's triangle are precisely the correct coefficients in the expansion of $(a+b)^n$. Thus, the structure of the triangle directly reflects the algebra of expanding powers.*

2.5.1 Consequences of the Binomial Theorem

By choosing the parameters appropriately, the binomial theorem immediately yields important identities for binomial coefficients. Two particularly well-known formulas arise by setting a and b equal to 1, respectively to 1 and -1 . At the same time, this shows how closely algebra and combinatorics are connected.

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The binomial theorem provides not only a formula for expanding $(a + b)^n$, but also several important identities for binomial coefficients. These become especially transparent when special values are substituted for a and b . Together with the combinatorial interpretation of binomial coefficients, this leads to two fundamental summation formulas.

Theorem 23. *For all $n \in \mathbb{N}_0$, one has*

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

Moreover, for all $n \in \mathbb{N}$, one has

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0.$$

Proof. We apply the binomial theorem.

Substituting $a = 1$ and $b = 1$ into

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

gives

$$(1 + 1)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} 1^k.$$

Since $1^{n-k} = 1$ and $1^k = 1$, it follows that

$$2^n = \sum_{k=0}^n \binom{n}{k}.$$

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This proves the first assertion.

Now substituting $a = 1$ and $b = -1$ gives

$$(1 + (-1))^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} (-1)^k.$$

Hence

$$0^n = \sum_{k=0}^n (-1)^k \binom{n}{k}.$$

For $n \in \mathbb{N}$, we have $0^n = 0$, and therefore

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0.$$

This proves the second assertion as well. □

Remark 24 (Combinatorial interpretation of the first identity). *The identity*

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

can also be understood directly from a combinatorial point of view. By the theorem on the number of k -element subsets, $\binom{n}{k}$ gives the number of subsets with exactly k elements of a set with n elements. Summing over all possible values of k therefore counts all subsets of an n -element set.

Since each of the n elements is either contained in a subset or not, there are exactly two choices for each element. Hence an n -element set has precisely

$$2^n$$

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subsets in total. Thus

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

is also immediately plausible from a combinatorial perspective.

2.6 The Geometric Series

A particularly important special case of sums arises when the summands are generated by repeatedly multiplying by the same number. Such series are called *geometric series*. They occur in many areas of mathematics, for example in estimates, in the study of limits, in interest calculations, and later also in power series.

Theorem 25 (Formula for the geometric series). *Let $a \in \mathbb{R}$ with $a \neq 1$, and let $n \in \mathbb{N}_0$. Then*

$$\sum_{k=0}^n a^k = \frac{1 - a^{n+1}}{1 - a}.$$

Proof. We prove the statement by complete induction on n .

Base case: For $n = 0$, we have

$$\sum_{k=0}^0 a^k = a^0 = 1.$$

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On the other hand,

$$\frac{1 - a^{0+1}}{1 - a} = \frac{1 - a}{1 - a} = 1.$$

Thus the statement is true for $n = 0$.

Induction hypothesis: Assume that, for some fixed but arbitrary $n \in \mathbb{N}_0$, the formula already holds, that is,

$$\sum_{k=0}^n a^k = \frac{1 - a^{n+1}}{1 - a}.$$

Induction step: We now prove the formula for $n + 1$. We have

$$\sum_{k=0}^{n+1} a^k = \sum_{k=0}^n a^k + a^{n+1}.$$

Using the induction hypothesis, it follows that

$$\sum_{k=0}^{n+1} a^k = \frac{1 - a^{n+1}}{1 - a} + a^{n+1}.$$

We put the terms over a common denominator:

$$\frac{1 - a^{n+1}}{1 - a} + a^{n+1} = \frac{1 - a^{n+1} + a^{n+1}(1 - a)}{1 - a}.$$

After expanding, we obtain

$$\frac{1 - a^{n+1} + a^{n+1} - a^{n+2}}{1 - a} = \frac{1 - a^{n+2}}{1 - a}.$$

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Hence

$$\sum_{k=0}^{n+1} a^k = \frac{1 - a^{n+2}}{1 - a}.$$

Thus the statement has also been proved for $n + 1$.

Therefore, the formula holds for all $n \in \mathbb{N}_0$. $\square$

Remark 26. *For $a = 1$, the formula is not defined in this form. In this case, however, we immediately have*

$$\sum_{k=0}^n 1^k = \sum_{k=0}^n 1 = n + 1.$$

The general formula illustrates particularly well how a closed form can arise from a simple repeated structure.

Later, the geometric series will be important, among other things, for limits, series, and estimates. It is therefore one of the fundamental tools of analysis.

The geometric series forms a fitting conclusion to this chapter. It shows once again how effective mathematical induction is as a proof principle and how closely it is connected with counting, summation, and the algebraic structure of the natural numbers. Thus the essential foundations have been laid on which the further study of analysis builds.

2.7 Exercises and Solutions

The following exercises revisit the central contents of this chapter. Each subsection contains exactly one exercise

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with its solution given immediately afterwards. In this way, the concepts and proof ideas can be reviewed step by step.

1.1 Proof by Mathematical Induction

Exercise 27. *Prove by mathematical induction:*

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \quad (n \in \mathbb{N}).$$

Solution 28. *We prove the statement by mathematical induction on n .*

For $n = 1$, we have

$$1 = 1^2.$$

Thus the assertion is true for the initial value.

Now assume that the statement holds for a fixed $n \in \mathbb{N}$, that is,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Then, for the next case, it follows that

$$1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = n^2 + (2n + 1).$$

Since

$$n^2 + 2n + 1 = (n + 1)^2$$

holds, we obtain

$$1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2.$$

Thus the assertion has been proved for $n + 1$. Hence the formula holds for all $n \in \mathbb{N}$.

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1.2 Summation Notation

Exercise 29. *Write out the sum*

$$\sum_{k=1}^4 (2k - 1)$$

explicitly and compute its value.

Solution 30. *The summation sign means that the terms for $k = 1, 2, 3, 4$ are added:*

$$\sum_{k=1}^4 (2k - 1) = 1 + 3 + 5 + 7.$$

Adding these numbers gives

$$1 + 3 + 5 + 7 = 16.$$

Therefore,

$$\sum_{k=1}^4 (2k - 1) = 16.$$

This also agrees with the general formula

$$\sum_{k=1}^n (2k - 1) = n^2,$$

since for $n = 4$ we have $4^2 = 16$.

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1.3 Product Notation and Factorials

Exercise 31. *Compute*

$$\prod_{k=1}^5 k$$

and explain its connection with the factorial.

Solution 32. *The product notation means that the factors for $k = 1, 2, 3, 4, 5$ are multiplied:*

$$\prod_{k=1}^5 k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$$

By definition, precisely this expression is the factorial of 5:

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$$

Therefore,

$$\prod_{k=1}^5 k = 5!.$$

1.4 The Binomial Coefficient

Exercise 33. *Compute*

$$\binom{5}{2}.$$

Then briefly explain the combinatorial meaning of the result.

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Solution 34. *Using the definition of the binomial coefficient, we obtain*

$$\binom{5}{2} = \frac{5!}{2!3!}.$$

Now

$$5! = 120, \quad 2! = 2, \quad 3! = 6,$$

so

$$\binom{5}{2} = \frac{120}{2 \cdot 6} = 10.$$

This means that from a set with 5 elements, one can choose exactly 10 different 2-element subsets when the order does not matter.

1.5 The Binomial Theorem

Exercise 35. *Expand $(x+y)^3$ using the binomial theorem.*

Solution 36. *For $n = 3$, the binomial theorem states that*

$$(x+y)^3 = \sum_{k=0}^3 \binom{3}{k} x^{3-k} y^k.$$

The binomial coefficients are

$$\binom{3}{0} = 1, \quad \binom{3}{1} = 3, \quad \binom{3}{2} = 3, \quad \binom{3}{3} = 1.$$

Therefore, we obtain

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

This is the complete expansion of $(x+y)^3$.

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1.5.1 Consequences of the Binomial Theorem

Exercise 37. *Show that*

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

Solution 38. *We substitute $a = 1$ and $b = 1$ into the binomial theorem. Then*

$$(1 + 1)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} 1^k.$$

Since every power of 1 is again 1, the right-hand side simplifies to

$$\sum_{k=0}^n \binom{n}{k}.$$

On the left-hand side, we have

$$(1 + 1)^n = 2^n.$$

Hence,

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

This formula also has a natural combinatorial interpretation: it counts all subsets of a set with n elements.

1.5.2 The Geometric Series

Exercise 39. *Show that for $a \neq 1$ and $n \in \mathbb{N}_0$:*

$$\sum_{k=0}^n a^k = \frac{1 - a^{n+1}}{1 - a}.$$

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Solution 40. We prove the formula by complete induction on n .

For $n = 0$, we have

$$\sum_{k=0}^0 a^k = a^0 = 1.$$

The right-hand side gives

$$\frac{1 - a^{0+1}}{1 - a} = \frac{1 - a}{1 - a} = 1.$$

Thus the statement is true for $n = 0$.

Now assume that the formula holds for some fixed $n \in \mathbb{N}_0$, that is,

$$\sum_{k=0}^n a^k = \frac{1 - a^{n+1}}{1 - a}.$$

Then for $n + 1$ we obtain:

$$\sum_{k=0}^{n+1} a^k = \sum_{k=0}^n a^k + a^{n+1}.$$

Using the induction hypothesis, it follows that

$$\sum_{k=0}^{n+1} a^k = \frac{1 - a^{n+1}}{1 - a} + a^{n+1}.$$

Bringing the terms to a common denominator gives

$$\frac{1 - a^{n+1} + a^{n+1}(1 - a)}{1 - a} = \frac{1 - a^{n+2}}{1 - a}.$$

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Hence the formula also holds for $n + 1$. Therefore it is proved for all $n \in \mathbb{N}_0$.

This formula is important because it transforms a long sum into a closed form. Later, it will play a central role in limits, series, and estimates.

Chapter 3

The Field Axioms of the Real Numbers

In the preceding sections, we have seen how mathematical statements can be formulated precisely and proved with the help of logical methods. We now turn to one of the central number systems of analysis, the real numbers $\mathbb{R}$.

The real numbers do not merely form a set on which one can add and multiply; they also possess a number of fundamental properties that describe these operations in a characteristic way. These properties are called the *field axioms*.

We first regard $\mathbb{R}$ as a set equipped with two operations

$$+ : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad (x, y) \mapsto x + y,$$

and

$$\cdot : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad (x, y) \mapsto x \cdot y.$$

CHAPTER 3. THE FIELD AXIOMS OF THE REAL NUMBERS

The first section is devoted to the axioms of addition.

3.1 The Axioms of Addition

Addition on the real numbers satisfies the following properties:

1. **Associative law:** For all $x, y, z \in \mathbb{R}$, one has

$$(x + y) + z = x + (y + z).$$

2. **Commutative law:** For all $x, y \in \mathbb{R}$, one has

$$x + y = y + x.$$

3. **Existence of an identity element:** There exists an element $0 \in \mathbb{R}$ with the property that, for all $x \in \mathbb{R}$,

$$x + 0 = x.$$

4. **Existence of additive inverses:** For every $x \in \mathbb{R}$, there exists an element $-x \in \mathbb{R}$ such that

$$x + (-x) = 0.$$

The identity element of addition is called *zero*; the inverse element belonging to x is called the *opposite number* or the *additive inverse* of x .

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3.1.1 Consequences of the Addition Axioms

From the addition axioms one can already derive several fundamental properties that will be used repeatedly throughout analysis. Many later calculations and proofs are based on these simple consequences.

1.

Theorem 41. *The zero element is uniquely determined.*

Proof. Let 0 and $0'$ be two neutral elements with respect to addition. Then

$$0 = 0 + 0'$$

by the neutrality of $0'$. On the other hand,

$$0 + 0' = 0'$$

by the neutrality of 0 . Hence $0 = 0'$. Thus the neutral element is uniquely determined. $\square$

2.

Theorem 42. *For every $x \in \mathbb{R}$, the additive inverse is uniquely determined.*

Proof. Let y and z be two additive inverses of x . Then

$$x + y = 0 \quad \text{and} \quad x + z = 0.$$

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We add z to both sides of the first equation. This gives

$$(x + y) + z = 0 + z.$$

By associativity and commutativity of addition, it follows that

$$(x + z) + y = z.$$

Since $x + z = 0$, we obtain $0 + y = z$, and therefore $y = z$. $\square$

3.

Theorem 43. *We have $-0 = 0$.*

Proof. By definition, -0 is an additive inverse of 0 . But since

$$0 + 0 = 0$$

also holds, 0 itself is an additive inverse of 0 . By the uniqueness of the additive inverse, it follows that

$$-0 = 0.$$

$\square$

4.

Theorem 44. *The equation $a + x = b$ has exactly one solution, namely $x = b - a$.*

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Proof. First we prove existence. If we set

$$x = b - a = b + (-a),$$

then, using associativity and commutativity of addition, we obtain

$$a + x = a + (b + (-a)) = (a + (-a)) + b = 0 + b = b.$$

Thus $x = b - a$ is a solution.

To prove uniqueness, let x be any solution of $a + x = b$. Then

$$a + x = b.$$

Adding $-a$ to both sides gives

$$(-a) + (a + x) = (-a) + b.$$

By associativity it follows that

$$((-a) + a) + x = (-a) + b,$$

hence

$$0 + x = (-a) + b.$$

Therefore

$$x = (-a) + b = b + (-a) = b - a.$$

Hence the solution is uniquely determined. $\square$

5.