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NONLINEAR ANALYSIS & DYNAMICAL SYSTEMS

Yuming QIN and Bin YANG

Attractors for Non-Classical Diffusion Equations and Kirchhoff Wave Equations

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Yuming QIN and Bin YANG

This book presents the latest research on global well-posedness including asymptotic behavior of solutions to some non-classical diffusion equations with fading memories, nonlocal terms or delays in several time-dependent spaces. The results collected in this book have been established by the authors and their collaborators over recent years.

This book has two distinguishing features. First, while there are many published works on non-classical diffusion equations in Sobolev spaces without time-dependent terms but few results in time-dependent spaces, this book fills this gap. Second, this book provides new results on the existence, regularity and upper semicontinuity of time-dependent global attractors, strong attractors, and pullback attractors in time-dependent spaces, as well as the ideas and methods for dealing with these problems that can be used in other related models.

Dr. QIN is a full-time professor, director of Institute of Nonlinear Sciences of Donghua University. His research interests are global (local) well-posedness of solutions and infinite dimensional dynamical systems for nonlinear evolutionary equations including fluid equations such as Navier–Stokes equations, MHD, and thermo (visco) elastic equations.

Bin YANG is a PhD student at College of Information Science and Technology, Donghua University. Her research interests are global well-posedness of solutions and infinite dimensional dynamical systems.

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To the memory of Yuming's father and mother, Zhenrong QIN and Xilan XIA

To Yuming's wife and son, Yu YIN and Jia QIN

To Bin's parents, Lijin YANG and Yongxia WANG

To Bin's sister, Jiabin YANG

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Preface

A dynamical system is mainly used to describe the state of the motion of dynamic systems. More precisely, it is used to study the law of the solutions to systems changing with time. When $t = 0, 1, 2, ...$, the system is called to be discrete and when $t \in \mathbb{R}$, the system is called to be continuous. If the dynamical system is classified according to whether the nonlinearity of the system depends on time, it can be divided into two categories. When the nonlinearity is independent of time, the system is called to be autonomous and when the nonlinearity is dependent on time, the system is called to be non-autonomous. Generally speaking, non-autonomous systems are more difficult to study than autonomous systems because their nonlinearities are more complex.

For autonomous systems, the long-time behavior of solutions is generally studied by investigating global attractors. A global attractor possesses invariance under the corresponding semigroup, compactness, and attractiveness. To determine the existence of a global attractor based on semigroup theory, three factors need to be verified: the continuity of the semigroup, dissipativity (the existence of an absorbing set), and compactness in some sense. It has a complex structure and is composed of bounded orbits, including equilibrium points, periodic orbits, quasi-periodic orbits, fractal dimensions, etc. It may not be a smooth manifold and has non-integer dimensions. At present, there are many monographs and papers on global attractors (exponential attractors) (see [8, 60, 69, 113, 115, 124], etc.). For non-autonomous systems, the global attractors are no longer applicable, thus many scholars turn to study more complex attractors such as pullback attractors, uniform attractors, etc. (see [24, 27, 34]).

For the past decades, many scholars have mostly focused on some common Sobolev spaces to discuss attractors. For concepts and definitions about Sobolev spaces, please refer to [52]. However, in the last decade, scholars no longer confined their research spaces to the conventional Sobolev spaces and began to study the time-dependent attractors in the space related to time. They considered their

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problems in the time-dependent space $\mathcal{H}_t(\mathbf{\Omega})$, which is endowed with the norms $\| u\|^2_{\mathcal{H}_t} = \| u\|^2_2 + \varepsilon(t) \| \nabla u\|^2_2$, for any $t \in \mathbb{R}$. If the time-dependent term $\varepsilon(t) \in C^1(\mathbb{R})$ is a decreasing bounded function and satisfies $\lim\limits_{t\to+\infty} \varepsilon(t) = 0$, and there exists a constant $L > 0$ such that $\sup\limits_{t\in\mathbb{R}}(|\varepsilon(t)| + |\varepsilon'(t)|) \leq L$, then these assumptions make it possible to study the existence of the time-dependent global attractors in $\mathcal{H}_t(\mathbf{\Omega})$.

Since the norm of the time-dependent space contains time-dependent term $\varepsilon(t)$, it is easy to see that the study of attractors in this space is more complex than the study of usual attractors in Sobolev spaces without time-dependent terms. It is necessary to carry out more complex calculations on the absorbing sets and asymptotic compactness to obtain the desired results. Although it is difficult to calculate and analyze, the existence of a time-dependent term broadens the previous research framework, which enables us to carry out research in models that are closer to physical reality and promotes the research process on the long-time behavior or well-posedness of solutions to dynamical systems, which is of great significance.

In addition, inspired by some works on time-dependent global attractors, we shall adjust the assumptions on the time-dependent function $\varepsilon(t)$, such as re-assuming its lower bound and monotonicity, and shall obtain some new works on the existence and regularity of pullback attractors and upper semicontinuity of pullback attractors and global attractors in some function spaces similar to $\mathcal{H}_t(\mathbf{\Omega})$, which are new attempts (see [111, 112, 144]).

When the density of the fluid inside is not uniform, the substance inside the object will be diffused from the high concentration of the place to the low concentration of the phenomenon, which is a common phenomenon in life, resulting in the diffusion of gases, liquid penetration, the diffusion of impurities in semiconductor materials and other related issues, involving such as aviation, meteorology, marine, biology, materials, and many other fields. Compared with classical diffusion equations, non-classical diffusion equations cover a wider range of physical phenomena, and the relevant mathematical theories and applications of these models have attracted much attention from scientists, and have great important scientific and applied value.

This book aims to present some latest results on attractors for non-classical diffusion equations and Kirchhoff wave equations in time-dependent spaces, which are divided into five parts. Firstly, the first part includes chapter 1, in which we shall systematically survey the works till 2023 on the time-dependent global attractors, and strong attractors and pullback attractors in the time-dependent spaces. Then, the second part includes chapter 2, where time-dependent global attractors for a class of reaction–diffusion equations are considered. To be more specific, chapter 2 will state the long-time behavior of weak solutions to the non-classical diffusion equations with fading memory when the nonlinear term satisfies critical exponential growth in the framework of weighted time-dependent spaces. In addition, the third part includes chapters 3–7, in which the existence of strong attractors and pullback attractors for non-classical reaction–diffusion equations will be investigated in several different time-dependent spaces. Chapter 3 will cover the long-time behavior

of solutions to the non-classical diffusion equation with fading memory when the nonlinear term fulfills the polynomial growth of arbitrary order. Chapter 4 will study the long-time behavior of solutions to the non-classical diffusion equations with nonlocal diffusion when the nonlinear term satisfies critical exponential growth. Chapter 5 will consider the existence and upper semicontinuity of pullback attractors for non-autonomous nonlocal diffusion equations with the nonlinear term containing a perturbation term. Chapter 6 will discuss the asymptotic behavior of solutions to non-autonomous diffusion equations with delay containing some hereditary characteristics and a nonlocal diffusion term, and chapter 7 will prove the existence and regularity of pullback attractors for non-classical non-autonomous diffusion equations with delay under a new framework. Then the fourth part includes chapter 8, in which we shall survey the existing works on Kirchhoff wave equations with strong damping in detail. Finally, chapter 9 is the last part, in which we shall prove the existence, regularity, and fractal dimension of global attractors for a Kirchhoff wave equation with a strong damping and a memory.

We sincerely wish that the readers will know the main ideas and essence of the basic theories and methods of attractors of several different reaction–diffusion equations and Kirchhoff wave equations in time-dependent spaces. We also wish that the readers can be stimulated by some ideas from this book and continue to undertake further study and research after having read this book.

We are grateful to Xiaoling CHEN for her contribution to chapters 1–3 and Hongli WANG for her contribution to chapter 8 and section 9.4. We also want to take this opportunity to thank all the people who are concerned about us including our teachers, colleagues, and collaborators, etc.

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Chapter 1

Survey on Attractors in Time-Dependent Spaces

In this chapter, we shall recall some results on the existence of attractors in time-dependent spaces and some useful basic definitions and properties of function spaces and attractors. The readers can easily find the details in the related literature, see, *e.g.*, Babin and Vishik [8], Berger [10], Caraballo and Cheban [14], Caraballo, Herrera-Cobos and Marín-Rubio [15], Chueshov [34], Chueshov and Lasiecka [37], Conti, Pata and Temam [43], Hu, Ren and Zhang [62], Liang and Wang [76], Lions [77], Ma, Wang and Liu [85], Ma, Wang and Xu [86], Meng and Liu [91], Meng, Wu and Zhao [92], Meng, Yang and Zhong [93], Peng, Shang and Zheng [99], Plinio, Duane and Temam [101], Robinson [113], Temam [124], Wang, Hu and Gao [127], Wang, Li and Qin [135], Wang and Ma [125], Woinowsky-Krieger [140], Zhu, Sun and Xie [165], Zhu, Xie and Zhou [167], etc.

1.1 Time-Dependent Global Attractors

1.1.1 Oscillation Equations

• In 2011, Plinio, Duane and Temam [101] first studied the time-dependent global attractors of the oscillation equations. The equations [101] studied are derived from the following equations

$$\partial_{tt}u(x,t)+H\partial_t u(x,t) - e^{-2Ht}\partial_{xx}u(x,t)+V'(u(x,t))=0, \quad x\in(0,1), t\in\mathbb{R}, \tag{1.1.1}$$

where V is a nonlinearity of polynomial growth satisfying natural dissipation conditions. The equation (1.1.1) is a Klein–Gordon equation with a given nonlinearity and has Robertson–Walker metric for scalar fields on manifolds, and Hubble constant $H>0$ for expanded domains. In [101], such an equation is called to be an oscillation equation.

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Assume that D is a subset of a Banach space X, with norm

$$\|D\|_X = \sup_{z\in D}\|z\|_X.$$

[101] studied the following oscillation equations in one-dimensional space with periodic boundary conditions:

$$\begin{cases} \partial_{tt}u(t) + H\partial_t u(t) + e^{-2Ht}Au(t) + \varphi(u(t)) = 0, \quad t \geq s, \\ u(s) = u_0 \in H^1_{\text{per}}(0,1), \partial_t u(s) = v_0 \in L^2(0,1). \end{cases} \tag{1.1.2}$$

Suppose the nonlinearity $V \in C^2(\mathbb{R})$ and $V(0) = 0$. Assume $\varphi = V'$ satisfying

(i) $\varphi(0) = 0$;
(ii) there exist constants $a_0, a_2 > 0, a_1, a_3 \geq 0, q \geq 2$ such that

$$a_0|y|^{q-2} - a_1 \leq \varphi'(y) \leq a_2|y|^{q-2} + a_3.$$

Moreover, when $q = 2$, assume $a_0 > a_1$.

The phase space studied here is the following Banach space

$$X_t = H^1_{\text{per}}(0,1) \times L^2(0,1),$$

with norm $\|(u,v)\|_{X_t} = e^{-Ht}\|A^{1/2}u\| + \|u\|_{L^q} + \|v\|$. Besides, for all fixed $t, s \in \mathbb{R}$, $\|\cdot\|_{X_s}$ and $\|\cdot\|_{X_t}$ are equivalent.

Under the above assumptions, [101] obtained the following well-posed result.

Theorem 1.1.1. *The problem* (1.1.1) *has a strong continuous process* $S(t,s) : X_s \to X_t$ *satisfying the following dissipation estimates: for all* $z \in X$,

$$\mathcal{E}_{X_t}(S(t,s)z) \leq K_0(\mathcal{E}_{X_s}(z))e^{-\mu(t-s)} + K_1, \quad \forall t \geq s,$$

where $K_1 = 4c_1^{-1}(c_0 + b_1)$, c_1, K_0, μ *are positive constants that only depend on physical parameters* H, a_2 *and* q.

After a simple dissipation estimation by using the energy method, the following result about absorbing sets was obtained in [101].

Theorem 1.1.2. *There exists a constant* $R_{\mathbb{A}} = R_{\mathbb{A}}(H, a_1) > 0$ *that makes*

$$\mathbb{A} = \{\mathbb{A}(t) = \{z \in X_t : \mathcal{E}_{X_t}(z) \leq R_{\mathbb{A}}\}\}$$

be an absorbing set of the process $S(\cdot,\cdot)$.

[101] constructed a time-dependent global attractor of the oscillation equation and it was the first paper to give a definition of it.

In order to verify the existence of the time-dependent global attractors, a two-parameter solution operator will be described by a family of maps acting on a one-parameter family of spaces X_t, which is named as a process.

Definition 1.1.3 ***(Process).*** *For all $t \in \mathbb{R}$, let X_t be a family of Banach spaces. A process is a two-parameter family of mappings $\{S(t,s) : X_s \to X_t\}_{s \le t}$ with properties*

(i) $S(t,t) = Id_{X_t}$;
(ii) $S(t,s) \in \mathcal{C}(X_s, X_t)$.

Moreover, if $S(t,s)$ further satisfies

(iii) $S(\tau,t)S(t,s) = S(\tau,s)$ for any $s \le t \le \tau$,

then it is a continuous process.

Definition 1.1.4 ***(Pullback-bounded family).*** *A family of subsets $\mathcal{B} = \{\mathcal{B}(t) \subset X_t\}_{t\in\mathbb{R}}$ is pullback-bounded if*

$$R(t) = \sup_{s\in(-\infty,t]} \|\mathcal{B}(s)\|_{X_s} < +\infty, \quad \forall t \in \mathbb{R},$$

i.e., the sets $\mathcal{B}(t) \subset X_t$ are bounded for all times $t \in \mathbb{R}$ and $\|\mathcal{B}(s)\|_{X_s}$ is bounded as $s \to -\infty$.

Definition 1.1.5 ***(Pullback absorber).*** *A pullback-bounded family $\mathbb{A} = \{\mathbb{A}(t)\}$ is called to be a pullback absorber if for every pullback-bounded family $\mathcal{B}$ and for every $t \in \mathbb{R}$, there exists a time $t_0 = t_0(t) \le t$ such that*

$$S(t,s)\mathcal{B}(s) \subset \mathbb{A}(t), \quad \forall s \le t_0.$$

Definition 1.1.6 ***(Time-dependent global attractors).*** *A family of compact subsets $\mathcal{A} = \{\mathcal{A}(t) \subset X_t\}_{t\in\mathbb{R}}$ is called to be a time-dependent global attractor if it fulfills the following properties*

(i) (invariance) $S(t,s)\mathcal{A}(s) = \mathcal{A}(t)$, for every $s \le t$;
(ii) (pullback attraction) for every pullback-bounded family $\mathcal{B}$ and every $t \in \mathbb{R}$,

$$\lim_{s\to-\infty} \operatorname{dist}_{X_t}(S(t,s)\mathcal{B}(s), \mathcal{A}(t)) = 0.$$

If property (ii) holds uniformly w.r.t. $t \in \mathbb{R}$, $\mathcal{A}$ is a uniform time-dependent global attractor.

Remark 1.1.7. *In general, conditions (i) and (ii) are not sufficient for uniqueness of the time-dependent attractor. However, if we require, in addition,*

(iii) $\mathcal{A}$ is a pullback-bounded family,

then there exists at most one family satisfying (i)–(iii), i.e., a pullback-bounded time-dependent global attractor is unique in the class of pullback-bounded families. Indeed, assume that $\mathcal{A}$ and $\mathcal{A}'$ are two pullback-bounded time-dependent global attractors for the process $S(\cdot,\cdot)$.

Fix $t \in \mathbb{R}$ and observe that (ii) implies

$$\operatorname{dist}_{X_t}\big(S(t,s)\mathcal{A}'(s), \mathcal{A}(t)\big) \to 0, \quad s \to -\infty. \tag{1.1.3}$$

But $S(t,s)\mathcal{A}'(s)=\mathcal{A}'(t)$, so that $\mathcal{A}'(t)\subset\mathcal{A}(t)$, since $\mathcal{A}(t)$ is closed. Exchanging the roles of $\mathcal{A}(t)$ and $\mathcal{A}'(t)$, we get the reverse inclusion as well, so that $\mathcal{A}(t)=\mathcal{A}'(t)$, and finally $\mathcal{A}=\mathcal{A}'$.

Definition 1.1.8 ***(Time-dependent ω-limit).*** *Given a family of sets $\mathcal{B}$, its time-dependent ω-limit is the family $\omega_{\mathcal{B}}=\{\omega_{\mathcal{B}}(t)\subset X_t\}_{t\in\mathbb{R}}$, where $\omega_{\mathcal{B}}(t)$ is defined as*

$$\omega_{\mathcal{B}}(t)=\bigcap_{\tau\leq t}\overline{\bigcup_{s\leq\tau}S(t,s)\mathcal{B}(s)}, \tag{1.1.4}$$

and the above closure is taken in X_t.

An equivalent characterization is the following

$$\begin{aligned}\omega_{\mathcal{B}}(t)=\{&z\in X_t:\exists\, s_n\to-\infty,\\ &z_n\in\mathcal{B}(s_n)\ with\ \|S(t,s_n)z_n-z\|_{X_t}\to 0\ as\ n\to+\infty\}.\end{aligned} \tag{1.1.5}$$

In the following main theorem, [101] proved the existence, uniqueness and regularity of the time-dependent global attractor and derived it has a finite fractal dimension.

Theorem 1.1.9. *For any $t\in\mathbb{R}$, the family $\mathcal{A}(t)=\omega_{\mathbb{A}}(t)$ is the unique time-dependent global attractor of the process $S(\cdot,\cdot)$ generated by problem (1.1.1). In addition, after introducing the family of Banach spaces*

$$Y_t=H^2_{\rm per}(0,1)\times H^1(0,1),\quad \|(u,v)\|_{Y_t}=e^{-Ht}\|Au\|+\|u\|_{L^q}+\|A^{1/2}v\|+\|v\|,$$

then we have

$$\mathcal{A}(t)\subset Y_t,\quad \|\mathcal{A}(t)\|^2_{Y_t}\leq h(t),\quad \forall t\in\mathbb{R},$$

where $h(t)$ is a continuous increasing function about t, depending on $R_{\mathbb{A}}$.

1.1.2 Wave Equations

• In 2013, Conti, Pata and Temam [43] constructed the basic framework of attractors in time-dependent space based on the pullback attraction of attractors and summarized the relevant definitions (such as uniformly bounded sets, time-dependent absorbing sets, pullback attraction, asymptotic compactness, time-dependent global attractors, invariance, etc.) and gave the following existence theorem of attractors.

Theorem 1.1.10 ***(Uniqueness).*** *Suppose that the process $U(t,\tau):X_\tau\to X_t, t\geq\tau\in\mathbb{R}$ acts on the time-dependent space X_t. If the process $U(t,\tau)$ is asymptotically compact, then it has a unique time-dependent global attractor.*

In particular, [43] found that when the process $U(t,\tau)$ is continuous or T-closed for any $T>0$, the invariance of its attractor can be directly obtained. Based on the above theoretical framework, [43] also studied the long-time dynamic behavior of the following model

$$\begin{cases} \varepsilon(t)u_{tt} + \alpha u_t - \Delta u + f(u) = g, \quad t > \tau, \\ u(x,t)|_{\partial\Omega} = 0, \\ u(x,\tau) = a(x), u_t(x,\tau) = b(x), \end{cases} \tag{1.1.6}$$

where $a, b : \Omega \to \mathbb{R}$ are known, α is a positive constant, the external force term $g(x) \in L^2(\Omega)$. [43] made the following assumptions:

($\mathbf{H_1}$): $\varepsilon(\cdot) \in C^1(\mathbb{R})$ is a monotonically decreasing bounded function satisfying

$$\lim_{t\to+\infty} \varepsilon(t) = 0,$$

and there exists a constant $L>0$ such that

$$\sup_{t\in R}[|\varepsilon(t)| + |\varepsilon'(t)|] \le L.$$

($\mathbf{H_2}$): The nonlinearity $f \in C^2(\mathbb{R}), f(0)=0$ and satisfies for any $s \in \mathbb{R}$,

$$\liminf_{|s|\to+\infty} \frac{f(s)}{s} > -\lambda_1,$$

$$|f''(s)| \le C(1+|s|),$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and C is a positive constant.

Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm of $L^2(\Omega)$, assume $A=-\Delta$ and $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. Assume $H_\sigma = D\big(A^{\frac{\sigma}{2}}\big)$ is a Hilbert space, and define its inner product and norm as

$$\langle\cdot,\cdot\rangle_s = \langle A^{\frac{s}{2}}\cdot, A^{\frac{s}{2}}\cdot\rangle, \|\cdot\|_s = \|A^{\frac{s}{2}}\cdot\|.$$

For $t\in\mathbb{R}, 0\le\sigma\le 2$, assume the time-dependent space $H_t^\sigma = H_{\sigma+1} \times H_\sigma$ and define its norm as

$$\|\{a,b\}\|_{H_t^\sigma}^2 = \|a\|_{\sigma+1}^2 + \varepsilon(t)\|b\|_\sigma^2.$$

When $\sigma=0$, we put $H_t^0 = H_t$. Then the compact embedding $H_t^\sigma \hookrightarrow H_t$ holds.

Based on the above time-dependent spaces, [43] used the decomposition operator method to obtain the following conclusions.

Theorem 1.1.11. *The problem* (1.1.16) *generates a strong continuous process* $U(t,\tau) : H_\tau \to H_t, t\ge\tau\in\mathbb{R}$, *and continuous with respect to the initial value, that is, for any* $z_i = (a_i, b_i) \in H_\tau, \|z_i(\tau)\|_{H_\tau} \le R (i=1,2)$, *a constant* $K\ge 0$, *the following inequality holds*

$$\|U(t,\tau)z_1 - U(t,\tau)z_2\|_{H_t} \le e^{K(t-\tau)}\|z_1(\tau) - z_2(\tau)\|_{H_\tau}, \quad \forall t\ge\tau.$$

Theorem 1.1.12. *There exists a constant* $R_0 \geq 0$ *such that* $\mathscr{B} = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$ *is a time-dependent absorbing set of the process* $U(t,\tau)$ *and for some* $I_0 \geq R_0$, *we have*

$$\sup_{z\in\mathbb{B}_\tau(R_0)}\left[\|U(t,\tau)z\|_{H_t}+\int_\tau^{+\infty}\|u_t(y)\|^2dy\right]\leq I_0,\quad \forall\tau\in\mathbb{R}.$$

Theorem 1.1.13. *The process* $U(t,\tau): H_\tau \to H_t, t\geq\tau\in\mathbb{R}$ *generated by* (1.1.6) *has an invariant time-dependent global attractor* $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$, *and* A_t *is bounded in* H_t^1, *the boundary is independent of* t.

• In 2014, Conti and Pata [42] studied the asymptotic structure of the semilinear wave equation in the bounded domain as follows

$$\varepsilon u_{tt} + \alpha u_t - \Delta u + f(u) = g, \tag{1.1.7}$$

where f is nonlinearity, g is an external force term, α is a strictly positive damping coefficient and $\varepsilon = \varepsilon(t)$ is a positive function that tends to infinity and zero gradually with t.

Then the equation

$$\alpha u_t - \Delta u + f(u) = g \tag{1.1.8}$$

can be regarded as the limit state of equation (1.1.7).

For problem (1.1.7), Conti, Pata and Temam [43] proved that the problem (1.1.7) can generate a process $U(t,\tau)$ acting on the time-dependent space family $\{Z_t\}$, and the process has an invariant time-dependent global attractor $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$. Meanwhile, it is known that (1.1.8) can generate a semigroup $S(t)$, and the semigroup has a global attractor A_∞. Therefore, [42] mainly discussed whether $A_t \to A_\infty$ when $t \to +\infty$.

In order to solve problem (1.1.8), [42] first constructed a time-dependent space

$$Z_t = H_0^1(\Omega)\times L^2(\Omega),$$

with the norm

$$\|(u,u_t)\|_{Z_t}^2 = \|u\|_{H_0^1}^2 + \varepsilon(t)\|u_t\|_{L^2}^2.$$

Then [42] briefly reviewed the process and the definition of attractors in the time-dependent space, described the time-dependent global attractor as a part of the complete bounded orbital set (CBT) with reference to the classical case, and proved the following two new theorems.

Theorem 1.1.14. *Suppose* $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$ *is a time-dependent global attractor of the process* $U(t,\tau)$, *if* $\mathscr{A}$ *is invariant, then*

$$A_t = \{z(t)\in Z_t : z \text{ is } CBT \text{ of } U(t,\tau)\}.$$

Theorem 1.1.15. *Suppose for any CBT sequence* $z_n = (x_n, y_n)$ *and any* $t_n \to +\infty$ *of the process* $U(t,\tau)$*, if there exists a CBT* $w : \mathbb{R} \to \mathscr{X}$ *of the semigroup* $S(t)$ *and* $s \in \mathbb{R}$ *that suppose*

$$\|x_n(s+t_n) - w(s)\|_{\mathscr{X}} \to 0 \quad as\, n \to +\infty,$$

then

$$\lim_{t\to+\infty} \delta_{\mathscr{X}}(\mathbf{\Pi}_t A_t, A_\infty) = 0.$$

Here, $\delta_{\mathscr{X}}$ stands for the Hausdorff semidistance.

Theorem 1.1.14 gives the structure of the time-dependent global attractor according to the completely bounded orbit of the system and theorem 1.1.15 gives sufficient conditions for $A_t \to A_\infty$ in a proper sense. Subsequently, [42] studied the relationship between A_t and A_∞ based on the theorems 1.1.14 and 1.1.15 in a bounded domain $\Omega \subset \mathbb{R}^3$. Under the same assumptions in [43], the following conclusions were obtained in [42].

Theorem 1.1.16. *Under the above assumptions, it follows that* $\lim_{t\to+\infty} \delta_{H^1}(\mathbf{\Pi}_t A_t, A_\infty) = 0$.

Finally, using energy estimation method, [42] further proved the following theorem about the regularity of attractors.

Theorem 1.1.17. *If* $\beta := \inf_{t\in\mathbb{R}}[2\alpha + \varepsilon'(t)] > 0$*, set*

$$\mathbf{\Pi}\mathscr{A} = \{u : \mathbb{R} \to H^1 \text{ such that } u = \mathbf{\Pi} z \text{ with } z \text{ is CBT of } U(t,\tau)\},$$

then

$$\sup_{u\in\mathbf{\Pi}\mathscr{A}} \sup_{t\in\mathbb{R}} \left[\|u_t(t)\|_1 + \sqrt{\varepsilon(t)}\|u_{tt}(t)\|\right] \le c.$$

• In 2016, Meng, Yang and Zhong [93] studied the time-dependent global attractor of the wave equation with nonlinear damping under the Dirichlet boundary condition on the time-dependent space and established a necessary, and sufficient condition for the existence of attractors in a time-dependent space, which is equivalent to the condition proved by Conti *et al.* in [43]. In addition, a technical method to verify the compactness of the process through the contraction function was also presented.

Suppose $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, [93] considered the following wave equation

$$\begin{cases} \varepsilon(t)u_{tt} + g(u_t) - \Delta u + \varphi(u) = f(x), \quad t > \tau, \tau \in \mathbb{R}, \\ u|_{\partial\Omega} = 0, \\ u(x,\tau) = u_0, u_t(x,\tau) = u_1, \end{cases} \tag{1.1.9}$$

where $u = u(x,t) : \Omega \times [\tau, +\infty) \to \mathbb{R}$ is an unknown function, $u_0, u_1 : \Omega \to \mathbb{R}$ is the given initial value and an external force term $f \in L^2(\Omega)$.

The function $\varepsilon(t)$ only depends on time, the damping term g is nonlinear, and the nonlinearity φ has a critical index.

Other assumptions in [93] about $\varepsilon(t)$, g and φ were as follows.

Suppose $\varepsilon \in C^1(\mathbb{R})$ is a bounded decreasing function and satisfies

$$\lim_{t\to+\infty} \varepsilon(t) = 0.$$

Particularly, there is a constant $L > 0$ such that

$$\sup_{t\in\mathbb{R}}[|\varepsilon(t)| + |\varepsilon'(t)|] \le L.$$

Assume $g \in C^1(\mathbb{R})$, $g(0) = 0$, g is strictly decreasing and satisfies

$$\liminf_{|s|\to+\infty} g'(s) > 0, \quad |g(s)| \le C_0(1 + |s|^p),$$

where $1 \le p < 5$.

Suppose $\varphi \in C^1(\mathbb{R})$ meets growth conditions

$$|\varphi'(s)| \le C\Big(1 + |s|^2\Big),$$

and dissipative conditions

$$\liminf_{|s|\to+\infty} \frac{\varphi(s)}{s} > -\lambda_1,$$

where $\lambda_1 > 0$ is the first eigenvalue of $A = -\Delta$ in $\mathcal{D}(A) = H^2(\Omega)\bigcap H_0^1(\Omega)$.

One of the main results of [93] is that the necessary and sufficient conditions for the existence and uniqueness of the time-dependent global attractor, which are as follows.

Theorem 1.1.18. *Let $U(\cdot,\cdot)$ is the process in Banach space $\{X_t\}_{t\in\mathbb{R}}$, then $U(\cdot,\cdot)$ has a time-dependent global attractor $\mathfrak{A}^* = \{A_t^*\}_{t\in\mathbb{R}}$ satisfying*

$$A_t^* = \bigcap_{s\le t}\bigcup_{\tau\le s} U(t,\tau)B_\tau,$$

only when

(i) $U(\cdot,\cdot)$ has a pullback absorbing family $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$;
(ii) $U(\cdot,\cdot)$ is pullback asymptotically compact.

The space in [93] is

$$\mathcal{H}_t = H_0^1(\Omega) \times L^2(\Omega),$$

with norm

$$\|\{u_0, u_1\}\|_{\mathcal{H}_t}^2 = \|\nabla u_0\|^2 + \varepsilon(t)\|u_1\|^2.$$

[93] also used the Galerkin method to obtain the existence and uniqueness of the weak solution of equation (1.1.9). The existence of the following time-dependent global attractors was obtained by the energy method.

Theorem 1.1.19. *The process* $U(t,\tau):\mathcal{H}_\tau\to\mathcal{H}_t$ *of problem* (1.1.9) *generates a time-dependent global attractor* $\mathfrak{A}=\{A_t\}_{t\in\mathbb{R}}$.

• In 2017, Meng and Liu [91] proved a new sufficient and necessary condition for the existence of a time-dependent global attractor and which was applied to below (1.1.10). [91] first reviewed the definition and properties of non-compact measurement in general spaces, and then constructed non-compact measurement $\gamma_t(X_t)$ in time-dependent spaces. Then a new compactness concept called ω_t-limit compactness was introduced for the process in time-dependent space, and the following theorem was proved.

Theorem 1.1.20. *Suppose* $U(t,\tau)$ *is a process in Banach space* $\{X_t\}_{t\in\mathbb{R}}$, *if* $U(t,\tau)$ *is* ω_t*-limit compact, then it is pullback asymptotically compact.*

Theorem 1.1.21. *Suppose* $U(t,\tau)$ *is a process in Banach space* $\{X_t\}_{t\in\mathbb{R}}$, *then* $U(t,\tau)$ *has a time-dependent global attractor* $\mathscr{A}=\{A_t\}_{t\in\mathbb{R}}$ *if the following assumptions hold*

(i) $U(t,\tau)$ *has a pullback absorbing set* $\mathscr{B}=\{B_t\}_{t\in\mathbb{R}}$;
(ii) $U(t,\tau)$ *is* ω_t*-limit compact.*

[91] also gave the definition of condition (C_t) and a method for verifying ω_t-limit compactness.

Definition 1.1.22 ***(Condition*** (C_t)***).*** *The process* $U(t,\tau)$ *is said to satisfy Condition* (C_t) *in time-dependent space if for any bounded set* B_τ *of* X_τ *and for any* $\varepsilon>0$, *there exists* $\tau_\varepsilon<t$, *and a finite-dimensional subspace* X_t' *of* X_t, *such that* $\{\|PU(t,\tau)B_\tau\|\}$ *is bounded and*

$$\|(I-P)U(t,\tau)x\|_{X_t}<\varepsilon\quad \text{for any}\ \tau\le\tau_\varepsilon, x\in B_\tau,$$

where $P:X_t\to X_t'$ *is a bounded projector. Specifically, when* X_t *is not a time-dependent space Condition* (C_t) *is referred to as Condition* (C).

Theorem 1.1.23. *If* $U(t,\tau)$ *satisfies the condition* (C_t) *in time-dependent space* $\{X_t\}_{t\in\mathbb{R}}$, *then the process* $U(t,\tau)$ *is* ω_t*-limit compact. If for any fixed* t, $\{X_t\}$ *is uniformly convex Banach space, the converse is also true.*

Combining theorems 1.1.20–1.1.23, the following theorem was obtained in [91].

Theorem 1.1.24. *Suppose* $U(t,\tau)$ *is a process in Banach space* $\{X_t\}_{t\in\mathbb{R}}$, *then* $U(t,\tau)$ *has a time-dependent global attractor* $\mathscr{A}=\{A_t\}_{t\in\mathbb{R}}$ *only when*

(i) $U(t,\tau)$ *has a pullback absorbing set* $\mathscr{B}=\{B_t\}_{t\in\mathbb{R}}$;
(ii) $U(t,\tau)$ *satisfies the condition* (C_t).

Finally, in strong space $V_t=(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$ with norm $\|(u,v)\|_{V_t}^2=\|u\|_{H^2(\Omega)\cap H_0^1(\Omega)}^2+\varepsilon(t)\|v\|_{H_0^1(\Omega)}^2$, the following problem was studied in [91]

$$\begin{cases}\varepsilon(t)u_{tt}+\alpha u_t-\Delta u+\varphi(u)=f(x), & t>\tau,\tau\in\mathbb{R},\\ u(x,t)|_{\partial\Omega}=0,\\ u(x,\tau)=u_0,u_t(x,\tau)=u_1,\end{cases}\tag{1.1.10}$$

where $u = u(x,t) : \Omega \times [\tau, +\infty) \to \mathbb{R}$ is unknown, $u_0, u_1 : \Omega \to \mathbb{R}$ is known, the damping coefficient α is a strictly positive constant, $f \in L^2(\Omega)$. Moreover, $\varepsilon(\cdot)$ is a monotonically decreasing bounded positive function and satisfies

$$\lim_{t\to+\infty} \varepsilon(t) = 0.$$

The nonlinearity $\varphi \in C^1(\mathbb{R})$ with $\varphi(0) = 0$ and satisfies

$$\liminf_{|s|\to+\infty} \frac{\varphi(s)}{s} > -\lambda_1,$$

$$|\varphi'(s)| \le C(1 + |s|^p), \quad 0 < p < 2,$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and C is a positive constant.

For the equation (1.1.10), by using theorem 1.1.24, the following conclusion was obtained in [91].

Theorem 1.1.25. *Under the assumptions given above, the process $U(t,\tau) : V_\tau \to V_t$, $t \ge \tau \in \mathbb{R}$ generated by problem (1.1.10) has an invariant time-dependent global attractor $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$.*

• Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^3$. For any $\tau \in \mathbb{R}$, in 2018, Ma, Wang and Liu [85] considered the following problem

$$\begin{cases} \varepsilon(t)u_{tt} + \alpha u_t - \beta\Delta u - \displaystyle\int_0^{+\infty} \mu(s)\Delta\eta^t(s)\,ds \\ \quad + f(u) = g(x), & \text{in } \Omega \times (\tau, +\infty), \\ u(x,t) = 0, & x \in \partial\Omega, t \in \mathbb{R}, \\ u(x,t) = u_0(x,t), u_t(x,t) = \partial_t u_0(x,t) & x \in \Omega, t \le \tau, \end{cases} \tag{1.1.11}$$

where $u = u(x,t) : \Omega \times [\tau, +\infty) \to \mathbb{R}$ is unknown, $u_0 : \Omega \times [-\infty, \tau) \to \mathbb{R}$ is known, $g(x) \in L^2(\Omega)$ is independent of time, μ is a summatable positive function, $\eta = \eta^t(x,s) = u(x,t) - u(x,t-s)$ for any $s \in \mathbb{R}^+$.

Suppose that $\varepsilon(\cdot) \in C^1(\mathbb{R})$ is a monotonically decreasing bounded function satisfying

$$\lim_{t\to+\infty} \varepsilon(t) = 0,$$

there is a constant $L > 0$ such that

$$\sup_{t\in\mathbb{R}} [|\varepsilon(t)| + |\varepsilon'(t)|] \le L.$$

The nonlinearity $f \in C^1(\mathbb{R}), f(0) = 0$ and statisfies for any $s \in \mathbb{R}$,

$$\liminf_{|s|\to+\infty} \frac{f(s)}{s} > -\lambda_1,$$

$$|f'(s)| \le C(1 + |s|^2),$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and C is a positive constant.

Suppose the memory term satisfies

$$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \int_0^{+\infty} \mu(s)\,ds = m_0 < +\infty,$$

$$\mu'(s) \leq -\rho\mu(s) \leq 0, \quad \forall s \geq 0,$$

where ρ is a positive constant.

By introducing new variable $\eta = \eta^t(x, s)$ and letting $\alpha = \beta = 1$, problem (1.1.11) can be written as

$$\begin{cases} \varepsilon(t)u_{tt} + u_t - \Delta u - \displaystyle\int_0^{+\infty} \mu(s)\Delta\eta^t(s)\,ds + f(u) = g(x), \\ \eta_t^t = -\eta_s^t + u, \end{cases} \tag{1.1.12}$$

with initial boundary value conditions

$$\begin{cases} u(x,t) = 0, & x \in \partial\Omega, t \geq \tau, \\ \eta^t(x,s) = 0, & (x,s) \in \partial\Omega \times \mathbb{R}^+, t \geq \tau, \\ u(x,\tau) = u_0(x), u_t(x,\tau) = u_1(x), & x \in \Omega, \\ \eta^\tau(x,s) = \eta^0(x,s), & (x,s) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{1.1.13}$$

where

$$\begin{cases} u_0(x) = u_0(x,\tau), \quad u_1(x) = \partial_t u_0(x,t)|_{t=\tau}, \\ \eta^0 = \eta^0(x,s) = u_0(x,\tau) - u_0(x,\tau - s). \end{cases}$$

In order to handle the function $\varepsilon(t)$ and the memory term, they constructed a time-dependent space

$$H_t^s = H^{s+1} \times H^s \times L_\mu^2(\mathbb{R}^+; H^{s+1}),$$

with the norm

$$\|z\|_{H_t^s}^2 = \|(u, u_t, \eta^t)\|_{H_t^s}^2 = \|u\|_{s+1}^2 + \varepsilon(t)\|u_t\|_s^2 + \|\eta^t\|_{\mu,s+1}^2.$$

Then by using the method in [43], the following theorems were obtained in [85].

Theorem 1.1.26. *Under the assumptions given above, the process* $U(t,\tau) : H_\tau \to H_t$, $t \geq \tau \in \mathbb{R}$ *generated by problems* (1.1.12) *and* (1.1.13) *has an invariant time-dependent global attractor* $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$.

Theorem 1.1.27. *The time-dependent global attractor* $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$ *in theorem* 1.1.26 *is bounded in* H_t^1 *with a bound independent of* t.

[85] used the ideas in [42] to study the relationship between the time-dependent global attractor $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$ and the global attractor generated by the limit state of equations (1.1.12) and (1.1.13) and obtained the following result.

Theorem 1.1.28. *Under the above assumptions,* $\lim\limits_{t\to+\infty}\delta_{H^1\times L^2_\mu(\mathbb{R}^+;H^1)}(\mathbf{\Pi}_t A_t, A_+\infty) = 0$ *holds.*

Finally, [85] further established the regularity result of the following attractors as follows.

Theorem 1.1.29. *Suppose* $\beta := \inf\limits_{t\in\mathbb{R}}[2+\varepsilon'(t)] > 0$ *holds,* $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$ *is an invariant time-dependent global attractor, then there exists a constant* $C = C(\mathscr{A}) > 0$ *such that*

$$\sup_{t\in\mathbb{R}}\sup_{(u,u_t,\eta^t)\in\mathscr{A}}\left[\|u_t(t)\|_1^2 + \varepsilon(t)\|u_{tt}(t)\|^2 + \|\eta_t^t\|_{\mu,1}^2\right] \le C.$$

• In 2023, Zhu, Sun and Xie [165] examined the long-time behavior of strong solutions for the following hyperbolic equation

$$\begin{cases} \varepsilon(t)u_{tt} - \Delta u_t - \Delta u + f(u) = g(x), & \text{in } \Omega\times(\tau,+\infty),\\ u(x,t) = 0, & \text{on } \partial\Omega\times(\tau,+\infty),\\ u(x,\tau) = u_\tau, u_t(x,\tau) = u'_\tau, & x\in\Omega, \end{cases} \tag{1.1.14}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $\tau\in\mathbb{R}$ is an initial time, $g(\cdot)\in L^2(\Omega)$ is a given external force, $\varepsilon(\cdot)\in C^1(\mathbb{R})$ is a bounded function satisfying

$$\varepsilon'(t) < 0 \quad \text{and} \quad \lim_{t\to+\infty}\varepsilon(t) = 0,$$

$$\sup_{t\in\mathbb{R}}[|\varepsilon(t)| + |\varepsilon'(t)|] \le L$$

for some constant $L > 0$, and the nonlinearity $f(u)$ is supposed to satisfy $f\in C^1(\mathbb{R},\mathbb{R})$ with $f(0) = 0$ and

$$\liminf_{|u|\to+\infty} f'(u) > -\lambda_1, \quad \forall u\in\mathbb{R},$$

here $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Define the Hilbert spaces $D(A^{\sigma/2}), \sigma\in\mathbb{R}$, with the inner products and norms in $L^2(\Omega)$ as

$$(\cdot,\cdot)_\sigma = (A^{\sigma/2}\cdot, A^{\sigma/2}\cdot) \quad \text{and} \quad \|\cdot\|_{D(A^{\sigma/2})} = \|A^{\sigma/2}\cdot\|_2,$$

where $A = -\Delta$ and $D(A) = H_0^1(\Omega)\cap H^2(\Omega)$.

Denote the time-dependent spaces $\mathcal{H}_t^\sigma (0\le\sigma\le 2)$ equipped with the norm

$$\|(u,u_t)\|_{\mathcal{H}_t^\sigma}^2 = \|A^{(1+\sigma)/2}u\|_2^2 + \varepsilon(t)\|A^{\sigma/2}u_t\|_2^2.$$